

KEY

LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS
MTH 201 - Calculus III

Exam-I, Fall 2012

Duration: 60 minutes

INSTRUCTIONS: This exam consists of 8 pages and 6 problems. Check that none is missing. Answer the questions in the space provided for each problem; if more space is needed, you may use the back pages. To receive full credits, you have to justify your answers.

Student's Name: _____

Student's ID: _____

Select your instructor and section

(a) Dr. Hamdan: (b) Dr. Issa MWF

(c) Dr. Issa T-TR (d) Dr. Touma

**Grading scheme
(Keep it empty)**

Question 1	/10
Question 2	/14
Question 3	/25
Question 4	/14
Question 5	/30
Question 6	/7
Total	/100

Problem 1: (10%) Evaluate the following integrals

$$\int \frac{e^x}{e^{2x}-1} dx$$

$$\text{let } u = e^x \quad du = e^x dx$$

$$\begin{aligned} \int \frac{e^x}{e^{2x}-1} dx &= \int \frac{du}{u^2-1} = \int \frac{du}{(u-1)(u+1)} = \frac{1}{2} \left[\int \frac{1}{u-1} du - \int \frac{1}{u+1} du \right] \\ &= \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C \\ &= \frac{1}{2} \ln \left| \frac{e^x-1}{e^x+1} \right| + C \end{aligned}$$

6 pts

$$\int \sinh(2 \ln t) dt$$

$$\int \sinh(2 \ln t) dt = \int \frac{e^{2 \ln t} - e^{-2 \ln t}}{2} dt = \int \frac{e^{\ln t^2} - e^{-\ln t^2}}{2} dt$$

$$= \frac{1}{2} \int \left\{ t^2 - \frac{1}{t^2} \right\} dt$$

$$= \frac{1}{2} \left\{ \frac{t^3}{3} - \frac{t^{-2+1}}{-2+1} \right\} + C$$

2 pts

Problem 2: (14%) Evaluate the following improper integrals

$$\int_2^{\infty} \frac{1}{\sqrt{t}(4+t)} dt \quad \text{Improper type (1), explicit}$$

$$u = \sqrt{t} \quad du = \frac{1}{2\sqrt{t}} dt = \frac{1}{2u} dt \Rightarrow dt = 2u du$$

$$\int \frac{1}{\sqrt{t}(4+t)} dt = \int \frac{2u du}{u(4+u^2)} = 2 \left[\frac{1}{2} \tan^{-1} u/2 \right] + C$$

$$= \tan^{-1} \sqrt{t}/2$$

$$\lim_{b \rightarrow \infty} \int_2^b \frac{1}{\sqrt{t}(4+t)} dt = \lim_{b \rightarrow \infty} \left[\tan^{-1} \sqrt{t}/2 \right]_2^b = \boxed{\pi/2 - \tan^{-1} \sqrt{2}/2}$$

7pts

$$\int_1^e \frac{1}{x \ln x} dx \quad \text{type (2), explicit}$$

$$\text{let } u = \ln x \quad du = 1/x dx$$

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln|u| = \ln|\ln x| + C$$

$$\lim_{b \rightarrow 1^+} \int_b^e \frac{1}{x \ln x} dx = \lim_{b \rightarrow 1^+} \ln|\ln x| \Big|_b^e = -(-\infty) = +\infty$$

7pts

Problem 3: (25%) Determine if each of the following improper integrals converges or diverges. Justify your answer and precise the test you used.

$$\int_0^{\infty} \frac{1}{x+e^x} dx \quad \text{type ① comparison}$$

$$\text{let } g(x) = \frac{1}{e^x}$$

$$\text{LCT with } g(x) = \frac{1}{e^x} \quad \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{1}{x+e^x} \cdot e^x = \lim_{x \rightarrow \infty} \frac{1}{1+x e^{-x}}$$

$$\int_0^{\infty} \frac{1}{e^x} dx = \int_0^{\infty} e^{-x} dx = -e^{-x} \Big|_0^{\infty} \xrightarrow{\text{limit}} 0 - (-e^0) \rightarrow 1$$

Test applies $\neq 0$
 $\neq \infty$

\Rightarrow Integral converges \Rightarrow original converges by LCT. 7pts ..

$$\int_1^{\infty} \frac{1}{\sqrt{x} e^{\sqrt{x}}} dx \quad \text{type ① explicit}$$

$$\text{let } u = \sqrt{x} \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{2}{\sqrt{x}} dx = du \cdot 2$$

$$\int \frac{1}{\sqrt{x} e^{\sqrt{x}}} dx = \int \frac{2 du}{e^u} = 2 \int e^{-u} du = -2e^{-u} = -2e^{-\sqrt{x}}$$

$$\int_1^{\infty} \frac{1}{\sqrt{x} e^{\sqrt{x}}} dx = \lim_{b \rightarrow \infty} -2e^{-\sqrt{x}} \Big|_1^b = 0 + 2e^{-1} \text{ Converges}$$

7pts

$$\int_2^{\infty} \frac{x}{\ln x} dx$$

Improper type ①

Divergence test

$\lim_{x \rightarrow \infty} \frac{x}{\ln x} \rightarrow \infty$ as $x \rightarrow \infty$

\Rightarrow Integral diverges

6 pts

$$\int_1^{\infty} \sin\left(\frac{1}{x}\right) dx. \text{ Hint: First find } \lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right)$$

Since $\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = 1$

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} \rightarrow 1$$

then $\int f(x) dx$ & $\int g(x) dx$ behave alike

by LCT \Rightarrow Since $\int_1^{\infty} \frac{1}{x} dx$ diverges p-int
 $p=1$

then $\int_1^{\infty} \sin\left(\frac{1}{x}\right) dx$ diverges

5 pts

Problem 4: (14%) Find the limit of each of the following sequences.

$$a_n = \left(\frac{n+4}{n+5} \right)^n$$

1^∞ is undetermined

$$a_n = \left[\frac{n(1+4/n)}{n(1+5/n)} \right]^n = \frac{(1+4/n)^n \rightarrow e^4}{(1+5/n)^n \rightarrow e^5} \quad \text{by rule \# - -}$$

$$\Rightarrow a_n \rightarrow \frac{e^4}{e^5} = 1/e$$

lopts

$$b_n = \frac{n2^n}{3^n} \quad \text{form } \infty \cdot 0 \text{ undetermined}$$

~~Can't do~~

~~$$n \left(\frac{2}{3} \right)^n = n e^{\ln \left(\frac{2}{3} \right)^n}$$

$$= n e^{n \ln \left(\frac{2}{3} \right)}$$

$$= n e^{n \ln \left(\frac{2}{3} \right)}$$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} n e^{n \ln \left(\frac{2}{3} \right)}$$

$$\lim_{n \rightarrow \infty} \frac{n e^{n \ln \left(\frac{2}{3} \right)}}{n} = \frac{e^{-\infty} \rightarrow 0}{1/n^2}$$~~

~~undetermined~~

$$b_n = e^{\ln(n(2/3)^n)}$$

$$= e^{\ln(n^n \cdot 2/3)^n} = e^{n \ln(n^n \cdot 2/3)}, \text{ as } n \rightarrow \infty$$

$\infty (-ve \#)$

$$\Rightarrow e \rightarrow 0$$

Remember $\ln \frac{2}{3} < 0$
 $\Rightarrow n \ln \left(\frac{2}{3} \right) \rightarrow -\infty$

Problem 5: (30%) Find the sum (if it exists) of each of the following infinite series

$$\sum_{n=1}^{\infty} \sqrt[n]{10n+5}$$

$a_n \rightarrow 1$ as $n \rightarrow \infty$ since $(10n+5)^{\frac{1}{n}}$
 $= e^{\ln(10n+5)^{\frac{1}{n}}}$
 $= e^{\frac{\ln(10n+5)}{n}} \rightarrow 0$ by l'Hop

This means

Series diverge by n^{th}

term test \Rightarrow Infinite sum
 is infinite (does not exist)

10 pts

$$\sum_{n=2}^{\infty} \frac{2^n + 5}{3^n}$$

2 Geometric series

$|r| < 1$ in each
 [careful does NOT
 start at 0]

$$\sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n + 5 \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n$$

10 pts

$$\frac{\left(\frac{2}{3}\right)^2}{1 - \frac{2}{3}} + \frac{5\left(\frac{1}{3}\right)^2}{1 - \frac{1}{3}}$$

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) \quad \text{Telescoping series}$$

$$\begin{aligned} S_n &= \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \dots + \ln\left(\frac{n}{n+1}\right) \\ &= \ln 1 - \ln 2 + \ln 2 - \ln 3 + \dots \\ &\quad + \dots - \ln n - \ln(n+1) \\ &= 0 - \ln(n+1) \end{aligned}$$

$$\lim_{n \rightarrow \infty} S_n = -\infty \Rightarrow \text{Series diverges}$$

Problem 6: (7%) Show that $0.\bar{9} = 1$

$$0.\bar{9} = 0.999\dots \quad 7 \text{ pts}$$

$$= 0.9 + 0.99 + 0.999 + \dots$$

$$= 0.9 \left(1 + 0.1 + 0.01 + \dots \right)$$

$$= 0.9 \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k \quad r = \frac{1}{10} < 1$$

$$= 0.9 \frac{1}{1 - 1/10} = \frac{0.9}{0.9} = 1$$

