MATH 201

Review sheet for EXAM 1.

I. Inverse trigonometric functions (Section 7.6)

(a) Their graph, Their limits ($\cos^{-1}$, $\sin^{-1}$, $\tan^{-1}$)

(b) How to differentiate them

(c) How to compute integrals involving inverse trig functions.

II. Hyperbolic functions (Section 7.7)

(a) Their definition, their limits

(b) How to simplify expressions involving hyperbolic functions (Always go back to definition with $e^x$..)

(c) How to differentiate them

(d) How to compute integrals with hyperbolic functions

III Techniques of Integration (Proper integrals 8.1, 8.4)

(1) Remember BASIC integration formulas like

- $\int [u(x)]^m u'(x)\,dx = \frac{u(x)^{m+1}}{m+1} + C \qquad m \neq -1$

- $\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$

- $\int e^{u(x)} u'(x)\,dx = e^{u(x)} + C$ , etc, ... See table @ the beginning of Chapter 8.

(2) Substitution: Whenever you see something as the derivative of something else

ex 1: $\int \frac{x^3}{1+x^4}\,dx$ $x^3$ is like the derivative of $x^4$ (up to a constant) so let $u = x^4$, ...

ex2: $\int \cos(x^2)\cdot x\,dx$ Let $u=x^2$, . . . .

(3) Integrals involving inverse trig functions:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}(u/a)+C$$

$$-\int \frac{du}{\sqrt{a^2-u^2}} = \cos^{-1}(u/a)+C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}(u/a)+C$$

$\downarrow$

these may not directly appear to you as such! You may have to do some effort to put them in this form.

examples: $\int \frac{dx}{\sqrt{4x-x^2}}$ $\longrightarrow$ complete the square here to reach $\sqrt{4-(x-2)^2}$, use substitution ... to get to $\int \frac{du}{\sqrt{a^2-u^2}}$

$\int \frac{dx}{9x^2+6x-12}$ $\longrightarrow$ also complete the square to reach $\int \frac{dx}{(3x-2)^2+2}$, use substitution to get to $\int \frac{du}{u^2+2}$ ...

$\int \frac{x}{4+9x^4}\,dx$ $\longrightarrow$ Substitution $u=3x^2$ .. to get to $\sim \int \frac{du}{4+u^2}$ etc, . . .

(4) Integration by parts:

For products of functions like $\int e^x \cos x$    $e^x$ + trig functions

$\int x^2 \cos x$    trig + polynomial

or for inverse trig function $\int \cos^{-1} x \, dx$   $(1 \cdot \cos^{-1} x)$

etc, ..

(5) Partial fractions:

For rational functions ($P/q$)

- Always simplify first, If deg(num) > deg(Denom) ⇒ divide   ex $\int \frac{x^2+2x+1}{x+1}\,dx = \int \frac{(x+1)^2}{(x+1)}\,dx = \int (x+1)\,dx$

- Try to factorize the denominator as much as possible ex: $\int \frac{dx}{x^2-2x-3} = \int \frac{dx}{(x-3)(x+1)} = \int \frac{A}{(x-3)} + \int \frac{B}{(x+1)}$ etc, ..

- Once simplified, look carefully at denominator; If you see a linear factor $(x-r) \rightsquigarrow \frac{A}{(x-r)}$

  If you see quadratic $ax^2+bx+c \rightsquigarrow \frac{Bx+C}{ax^2+bx+c}$

  (Review Notes)

ex $\int \frac{dx}{\underbrace{(x-1)^2}_{\text{repeated linear}}\underbrace{(x^2+3)}_{\text{Quadratic}}} = \int \frac{A\,dx}{(x-1)} + \int \frac{B\,dx}{(x-1)^2} + \int \frac{Cx+D\,dx}{(x^2+3)}$

# IV Techniques of integration (8.7)
## IMPROPER INTEGRALS

$\int \underbrace{f(x)}_{\text{integrand}} dx$ with SOME PROBLEM !
so always write down corresponding limits.

• Identify the problem first :

(A) If the integrand has no asymptotes in the domain, but the domain is infinite

ex $\displaystyle\int_8^\infty \frac{dx}{(x-7)(x-6)}$ , $\displaystyle\int_0^\infty \frac{e^x}{\sqrt{1+x}}$ , $\displaystyle\int_1^\infty \frac{x}{\sqrt{x^3+1}}dx$ ...

then (1) Check if $f \to \infty$ , conclude integral diverges

ex $\displaystyle\int_0^\infty \frac{e^x}{\sqrt{1+x}}dx$ $\left(\lim_{x\to\infty} \frac{e^x}{\sqrt{1+x}} = \infty\right)$

(2) Check if you can explicity integrate

ex $\displaystyle\int_1^\infty \frac{\ln x}{x}dx$ , then integrate and evaluate limit

(3) If you can't integrate think of comparing with simpler integrals like $\int e^x$, $\int e^{-x}$, $\int \frac{1}{x^m}$ ...

Here, first make sure $f \geq 0$ , if not use the fact that $f$ could be even or odd

for $f$ even $\displaystyle\int_{-\infty}^{\infty} f = 2\int_0^\infty f$

for $f$ odd $\displaystyle\int_{-\infty}^{0} f = -\int_0^\infty f$

ex: $\int_{-\infty}^{0} \frac{x}{1+x^2}\,dx = -\int_{0}^{\infty} \frac{x}{1+x^2}\,dx$ since integrand is ODD

Now for comparison, use DCT or LCT

→ DCT applies when the comparison is DIRECT and easy. for ex:

- $\int_1^\infty \frac{\sin x}{x^2}\,dx$ Since $\sin x \le 1$

  $\Rightarrow \underbrace{\int_1^\infty \frac{\sin x\,dx}{x^2}}_{\text{Conv}} \le \underbrace{\int_1^\infty \frac{1}{x^2}}_{\text{Conv}}$

- $\int_1^\infty \frac{\cos x+3}{x^{1/2}}\,dx \Rightarrow \overbrace{\int_1^\infty \frac{\cos x+3\,dx}{x^{1/2}}}^{\text{Div}} \ge \overbrace{\int_1^\infty \frac{2}{x^{1/2}}\,dx}^{\text{Div}}$

  since $\cos x \ge -1$
  $\cos x + 3 \ge 2$

- $\int_1^\infty \frac{\ln x}{x+\sqrt{x-1}}\,dx$

  $\sqrt{x-1} < x$

  $x+\sqrt{x-1} < 2x$

  $\Rightarrow \underbrace{\int \frac{\ln x\,dx}{x+\sqrt{x-1}}}_{\text{Div}} > \frac{1}{2}\underbrace{\int \frac{\ln x}{x}\,dx}_{\text{Div}}$

- $\int_1^\infty e^{-x}/x^3\,dx$

  $e^{-x} \le e^{-1}$

  $\Rightarrow \underbrace{\int_1^\infty \frac{e^{-x}}{x^3}\,dx}_{\text{conv}} \le \underbrace{\int_1^\infty \frac{e^{-1}}{x^3}\,dx}_{\text{Conv}}$

$\rightsquigarrow$ LCT applies when the comparison is not so direct. this usually applies when we can analyze <u>rates</u> of functions at infinity (ex $\ln x$ slower than $\sqrt{x}$
$\sqrt{x}$ slower than $x$
$e^x$ faster than $x^p$ ...)

Here we have to make sure we choose $g$ such that we get $\lim f/g$ = finite $> 0$
$0 < L < \infty$

<u>examples:</u>

- $$\int_1^\infty \frac{x}{\sqrt{x^3+1}}\,dx \quad \sim \frac{x}{\sqrt{x^3}} = \frac{x}{x^{3/2}} = \frac{1}{x^{1/2}}$$

so if $f = \dfrac{x}{\sqrt{x^3+3}}$, $g = \dfrac{1}{x^{1/2}}$

$$\lim_{x\to\infty} f/g = \frac{x^{3/2}}{\sqrt{x^3+3}} = 1 \quad \text{(Hopital)}$$

$\Rightarrow$ Since $\int g = \int \frac{1}{x^{1/2}}$ diverges
original diverges

- $$\int_1^\infty \frac{\sqrt{x}}{1+x^2}\,dx \quad \sim \frac{x^{\frac{1}{2}}}{x^2} = \frac{1}{x^{1.5}} \leftarrow \text{choose as } g$$

continue, ...

See all examples of LCT from Sample Exam ¡

(B) The integrand has a problem (asymptote) @ a point in the domain, then identify it and split & write limits accordingly

Ex: • $\int_1^\infty \frac{1}{\sqrt{x^2-1}}\,dx = \lim_{b\to 1^-} \int_b^\infty \frac{1}{\sqrt{x^2-1}}\,dx$

↳ problem at 1

• $\int_{-1}^{1} \frac{1}{x^{2/3}}\,dx = \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^{2/3}}\,dx + \lim_{c\to 0^+} \int_c^1 \frac{1}{x^{2/3}}\,dx$

↳ problem at 0

Sometimes, the problem is not so clear, we have to do some effort to see it:

Ex • $\int_1^4 \frac{dx}{x^2-2x+1}$ ← have to factorize

$\int_1^4 \frac{dx}{(x-1)^2} = \lim_{b\to 1^+} \int_b^4 \frac{dx}{(x-1)^2}$

Then once we write the limits, we integrate (using proper integration techniques in III) and evaluate the limits!

# V Convergence or Divergence of Sequences:

- $\{a_n\}$ converges if there's a finite number $L$ such that $\lim_{n\to\infty} a_n = L$

If no such number exists then we say $\{a_n\}$ diverges

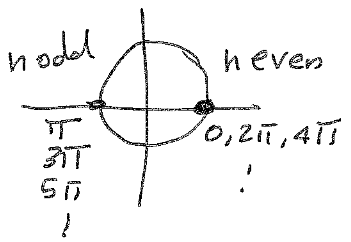
- there are many ways in which the sequence can diverge. for ex $\lim_{n\to\infty} a_n = +\infty$ (it blows up)

ex $a_n = \sqrt{n}$

or it can oscillate (not decide on a value)

for ex 1) $a_n = \{-1\}^n \nearrow +1 \to n$ even, $\searrow -1 \to n$ odd

or 2) $a_n = \cos(n\pi) \nearrow +1$ $n$ even, $\searrow -1$ $n$ odd




3) $a_n = \cos(n\pi)\cdot 3 \nearrow +3$, $\searrow -3$

4) $(-1)^n \cdot n \nearrow +\infty$ even, $\searrow -\infty$ odd

# Techniques to compute limits



- You have to know some ex $\frac{1}{n} \to 0$
  see table in section 10.1

- L'Hopital's rule for indeterminate forms
  like $\infty/\infty$, $0/0$, $\infty \cdot 0 \rightsquigarrow \frac{0}{1/\infty}$ or $\frac{\infty}{1/\infty}$

- Sandwich theorem : forms like $\frac{(-1)^n}{n!}$, $\frac{\cos n\pi}{n^2}$, ...
  or whenever we can do such comparisons.

- If we have powers, it's a good idea to
  use $e^{\ln}$ then l'Hopital's rule

  Ex $\sqrt[n]{2n} = (2n)^{1/n} = e^{\ln(2n)^{1/n}}$

  $= e^{\frac{1}{n}\ln(2n)}$

  $$\lim_{n\to\infty} \frac{\ln(2n)}{n} \underset{\substack{\uparrow \\ \text{Hopital's}}}{=} \lim_{n\to\infty} \frac{1/2n \cdot 2}{1} = 0$$

  $$\Rightarrow \lim_{n\to\infty} \sqrt[n]{2n} = e^0 = 1$$

- If f continuous and $\{a_n\} \to L$ then
  $f(a_n) \longrightarrow f(L)$ ex

  $$\lim_{n\to\infty} \sqrt{\frac{n+1}{n-1}} = \sqrt{1} = 1$$

GOOD LUCK !!