Test 2: MTH 304

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Duration: 150 minutes

Name (Last, First): ______________________________

Student number: ______________________________

Solution

For marker's use only	
Problem	Score
1	/15
2	/24
3	/36
4	/25
Bonus	7
Total	/105

Problem 1 [15 points] Solve the following ODE using the method of Undetermined coefficients.

$$y'' - 6y' + 9y = e^{3x}$$

$y'' - 6y' + 9y = 0$

$\lambda^2 - 6\lambda + 9 = 0 \Longrightarrow$ $\Delta = 36 - 36 = 0$

$= (\lambda - 3)^2$

$\lambda_1 = \lambda_2 = 3$

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$\Rightarrow y_1 = c\, e^{3t}$ is a solution $y_2 = x e^{3t}$ is a solution

The general solution is $y_h = c_1 y_1 + c_2 y_2 = (c_1 + c_2 x)\, e^{3x}$

$y_p = c x^2 e^{3x}$ by the Modification Rule

$y_p'' - 6y_p' + 9y_p = e^{3t}$

$y_p' = c\,2x e^{3x} + 3c x^2 e^{3x}$

$y_p'' = 2c e^{3x} + 2cx \times 3 e^{3x} + 6cx e^{3x} + 9cx^2 e^{3x}$

10

$\Rightarrow 2ce^{3x} + \cancel{12cxe^{3x}} + \cancel{9cx^2e^{3x}} - \cancel{12cxe^{3x}} - \cancel{18cx^2e^{3x}} + \cancel{9cx^2e^{3x}} = e^{3t}$

$\Rightarrow 2c = 1 \Rightarrow c = 1/2$

thus the general solution is $y = (c_1 + c_2 x)\, e^{3x} + \frac{1}{2} x^2 e^{3x}$

Problem 2 [24 points] The following questions are independent.

1. Find The Laplace Transform of $f(t) = \frac{1}{2\beta}(\sin\beta t + \beta t \cos\beta t)$.

$\mathcal{L}(f) = \frac{1}{2\beta} \mathcal{L}(\sin \beta t) + \beta \mathcal{L}(t \cos \beta t)$

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$= \frac{1}{2\beta} \frac{\beta}{s^2+\beta^2} + \frac{\beta}{2\beta} \frac{s^2 - \beta^2}{(s^2+\beta^2)^2}$

$= \frac{1}{2} \frac{1}{s^2+\beta^2} + \frac{1}{2} \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$

$$= \frac{1}{2} \frac{2s^2}{(s^2+\beta^2)^2} = \frac{s^2}{(s^2+\beta^2)^2}$$

2. Solve the following ODES using Laplace transforms.

(a) $f(t) + \int_0^t f(\tau)d\tau = 1$

$$\mathcal{L}(f) + \mathcal{L}(f) \times \mathcal{L}(1) = \mathcal{L}(1)$$

$$\Rightarrow \mathcal{L}(f)\left(1 + \frac{1}{s}\right) = \frac{1}{s}$$

$$\Rightarrow \mathcal{L}(f)\left(\frac{s+1}{s}\right) = \frac{1}{s} \Rightarrow \mathcal{L}(f) = \frac{1}{s+1}$$

$$\Rightarrow f(t) = e^{-t}$$

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(b) $y' + 4y = 3sin(t-5)u(t-5)$ with $y(0) = 1$ and $y'(0) = 0$.

$$\mathcal{L}(y'') + 4\mathcal{L}(y) = \mathcal{L}(3\sin(t-5)u(t-5))$$

$$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) + 4\mathcal{L}(y) = \frac{3e^{-5s}}{s^2+1}$$

$$s^2\mathcal{L}(y) - s + 4\mathcal{L}(y)$$

$$\Rightarrow \mathcal{L}(y) = \frac{3e^{-5s}}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}$$

$$\frac{3e^{-5s}}{(s^2+1)(s^2+4)} = \frac{e^{-5s}}{s^2+1} - \frac{e^{-5s}}{s^2+4}$$

$$\Rightarrow y = \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{e^{-5s}}{s^2+1}\right) - \mathcal{L}^{-1}\left(\frac{e^{-5s}}{s^2+4}\right)$$

$$= \cos 2t + u(t-5)\sin(t-5) - \frac{1}{2}u(t-5)\sin(2(t-5))$$

Problem 3 [36 points]. Find the inverse Laplace transform of each of the following.

1. $F(s) = \frac{1}{s^2+4s+13}$.

$\Delta = 16 - 4 \times 13 < 0$

$$= \frac{1}{(s+2)^2 + 9} \Rightarrow f(t) = \frac{1}{3} \sin 3t \left(e^{-2t}\right)$$

S

2. $F(s) = \frac{1}{s^2+9} e^{\frac{-\pi s}{2}}$.

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$$f(t) = \frac{1}{3} \sin 3(t - \pi/2)\, u(t - \pi/2)$$

3. $F(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{-16}{5}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2} + \frac{1}{30}\frac{1}{s+4}$$

$$\Rightarrow \quad f(t) = \frac{-16}{5} e^{t} + \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t}$$

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4. $F(s) = \frac{4s}{(s^2+4)^2}$.

$\mathcal{L}^{-1}(F(s)) = t \sin 2t$

5

5. $F(s) = \frac{s}{(s+1)^2}$

$= \frac{s+1-1}{(s+1)^2} = \frac{1}{s+1} \quad \frac{-1}{(s+1)^2}$ 5

$\Rightarrow f(t) = e^{-t} - te^{-t} = e^{-t}(1-t)$

6. $F(s) = \frac{1}{s(s^2+5s+6)}$ (Using convolution)

$$\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) * \mathcal{L}^{-1}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) = 1 * \left(e^{-2t} - e^{-3t}\right)$$

$$= \int_0^t 1\left(e^{-2\tau} - e^{-3\tau}\right) d\tau$$

5

$$= -\frac{1}{2}e^{-2\tau} + \frac{1}{3}e^{-3\tau} \Big|_0^t$$

$$= -\frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} - \left(-\frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}$$

7. $F(s) = \frac{s+1}{s^3+9s}$. (Using Laplace Transform of integral)

$$= \frac{s+1}{s(s^2+9)} \quad . \quad \mathcal{L}^{-1}\left(\frac{s+1}{s^2+9}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right)$$

6

$$= \cos 3t + \frac{1}{3}\sin 3t$$

$$\mathcal{L}^{-1}(F(s)) = \int_0^t \left(\cos 3u + \frac{1}{3}\sin 3u\right) du$$

$$= \left(\frac{1}{3}\sin 3u - \frac{1}{9}\cos 3u\right)\Big|_0^t$$

$$= \frac{1}{3}\sin 3t - \frac{1}{9}\cos 3t + \frac{1}{9}$$

Problem 4 [25 points]

1. Determine a system of differential equations that describes the currents $i_1(t)$ and $i_2(t)$ in the electrical network shown in the figure below.

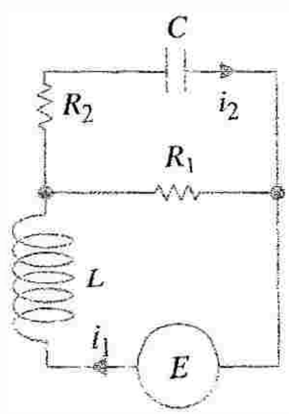

Figure 1

2. Using Laplace Transforms, Find the currents $i_1(t)$ and $i_2(t)$ in the network with $L = 1\ H$, $C = 2\ F$, $R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$, $E(t) = 1\ V$ and $i(0) = 0$, $i'(0) = 0$.

1) $\boxed{E(t) = L i_1' + R_1 (i_1 - i_2)}$

$$0 = R_2 i_2 + \frac{1}{C}\int i_2(t)dt + R_1 (i_2 - i_1)$$

$$0 = R_2 i_2' + \frac{1}{C} i_2(t) + R_1 (i_2' - i_1')$$

$$\boxed{0 = (R_1 C + R_2 C) i_2' + i_2 - R_1 C i_1'}$$

2) $1 = i_1' + i_1 - i_2$

$$0 = 6 i_2' + i_2 - 2 i_1'$$

$$\frac{1}{s} = sY_1 + Y_1 - Y_2$$

$$0 = 6sY_2 + Y_2 - 2sY_1$$

$$(s+1)Y_1 = Y_2 + \frac{1}{s} \quad (1)$$

$$(6s+1)Y_2 = 2sY_1 \quad (2)$$

$$\Rightarrow Y_2 = \frac{2s}{6s+1} Y_1$$

(1): $(s+1)Y_1 = \frac{2s}{6s+1} Y_1 + \frac{1}{s}$

$$(s+1)(6s+1) - 2sY_1 = \frac{6s+1}{s}$$

$$\Rightarrow (3s+1)(2s+1)Y_1 = \frac{6s+1}{s}$$

$$\Rightarrow Y_1 = \frac{6s+1}{s(3s+1)(2s+1)} = \frac{1}{s} + \frac{9}{3s+1} - \frac{8}{2s+1}$$

$$\Rightarrow y_1 = +3e^{-\frac{1}{3}t} - 4e^{-\frac{1}{2}t} + 1$$

$$i_2 = i_1 - i_1' - 1 = t + 4e^{-\frac{1}{3}t} - 6e^{-\frac{1}{2}t} - 2$$

$$\cancel{1} + 3e^{-1/3 t} - 4e^{-1/2 t} - \left(-e^{-1/3 t} + 2e^{-1/2 t}\right) \cancel{-1}$$

$$= 4e^{-1/3 t} - 2e^{-1/2 t}$$

Bonus: [7 points] Find the inverse Laplace Transform of the following function $F(s) = \frac{s}{(s^2-9)^2}$. Hint: $(\frac{1}{s^2-9})' = \frac{-2s}{(s^2-9)^2}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-9}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-3)(s+3)}\right) = \mathcal{L}^{-1}\left(\frac{1}{6}\frac{1}{s-3} - \frac{1}{6}\frac{1}{s+3}\right)$$

$$= \frac{1}{6}\left(e^{3t} - e^{-3t}\right) = f(t) = \frac{1}{3}\sinh 3t$$

$$\mathcal{L}(tf(t)) = -\left(\frac{1}{s^2-9}\right)' = \frac{2s}{(s^2-9)^2} \Rightarrow \mathcal{L}^{-1}\left(\frac{2s}{(s^2-9)^2}\right) = \frac{t}{3}\left(\frac{e^{3t}-e^{-3t}}{2}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{s}{(s^2-9)^2}\right) = \frac{t}{6}\left(\frac{e^{3t}-e^{-3t}}{2}\right) = \frac{t}{6}\sinh 3t$$