

LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS
MTH 304 – DIFFERENTIAL EQUATIONS
EXAM 1 – SPRING 2014

NAME:

KEY.

ID #:

THIS EXAM CONSISTS OF 9 PAGES AND 6 PROBLEMS. THE LAST PAGE MAY BE USED FOR SCRAP WORK. ANSWER THE QUESTIONS IN THE SPACES PROVIDED; IF MORE SPACE IS NEEDED, USE THE BACK OF THE PAGES. THE TIME OF THE EXAM IS 75 MINUTES.

NB. If you wish your exam graded by tomorrow (deadline for WI), tick the box

Question Number	Grade
1. (28%)	
2. (14%)	
3. (13%)	
4. (12%)	
5. (18%)	
6. (15%)	
TOTAL	

1. Identify the following differential equations as either **separable**, **linear**, or **exact**, then solve each one **explicitly** if possible. In case the problem is an **initial-value** one, find the **particular solution**.

a. (8%) $t \frac{dy}{dt} + y = t^2 e^3, t > 0.$

Divide by t : $\frac{dy}{dt} + \frac{1}{t}y = te^{t^3}; t > 0.$

This is linear: $p(t) = \frac{1}{t}$; $q(t) = te^{t^3}$

Integrating factor $\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t.$

$$\therefore y = \frac{1}{t} \int t \cdot te^{t^3} dt = \frac{1}{t} \left[\frac{1}{3} e^{t^3} + C \right]$$

$$= \frac{1}{3t} e^{t^3} + \frac{C}{t}.$$

b. (8%) $e^t y \frac{dy}{dt} = e^{-y} + e^{-2t-y}.$

$$e^t y \frac{dy}{dt} = e^{-y} (1 + e^{-2t}) \therefore \text{separable}$$

$$\therefore y e^y dy = e^{-t} (1 + e^{-2t}) dt$$

$$\int y e^y dy = \int (e^{-t} + e^{-3t}) dt = -e^{-t} - \frac{1}{3} e^{-3t} + C$$

$$\downarrow$$

$$u = y \rightarrow du = dy$$

$$dv = e^y \rightarrow v = e^y$$

$$\therefore \int y e^y dy = y e^y - \int e^y dy = y e^y - e^y$$

\therefore Implicit family of solutions is:

$$y e^y - e^y = -e^{-t} - \frac{1}{3} e^{-3t} + C.$$

c. (10%) $(\cos x - x \sin x + y^2) + 2xy \frac{dy}{dx} = 0; y(\pi) = 1.$

$$\underbrace{}_{M(x,y)} + \underbrace{}_{N(x,y)} \frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = 2y; \quad \frac{\partial N}{\partial x} = 2y \quad \therefore \text{exact.}$$

Solutions: $\Phi(x,y) = C$, where

$$\frac{\partial \Phi}{\partial x} = M; \quad \frac{\partial \Phi}{\partial y} = N.$$

$$\frac{\partial \Phi}{\partial y} = N \Rightarrow \Phi(x,y) = \int N(x,y) dy = \int 2xy dy = xy^2 + C(x).$$

$$\frac{\partial \Phi}{\partial x} = \cos x - x \sin x + y^2 \Rightarrow y^2 + C'(x)$$

$$\therefore C'(x) = \cos x - x \sin x$$

$$\therefore C(x) = \int (\cos x - x \sin x) dx$$

$$= \sin x - \underbrace{\int x \sin x dx}$$

$$u = x \rightarrow du = dx$$

$$dv = \sin x \rightarrow v = -\cos x$$

$$\therefore C(x) = \sin x - [-x \cos x + \int \cos x dx]$$

$$= \sin x + x \cos x - (\sin x)$$

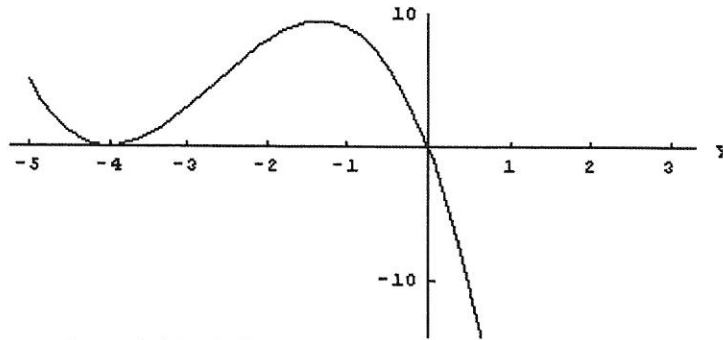
$$= x \cos x.$$

\therefore Family of solutions is: $xy^2 + x \cos x = C$

But $y(\pi) = 1 \Rightarrow \pi + \pi \cos(\pi) = C \Rightarrow \pi - \pi = 0$

\therefore family of solutions is: $\boxed{xy^2 + x \cos x = 0.}$

2. (14%) Consider the **autonomous** differential equation $\frac{dy}{dt} = f(y)$, where the graph of $f(y)$ is given below:

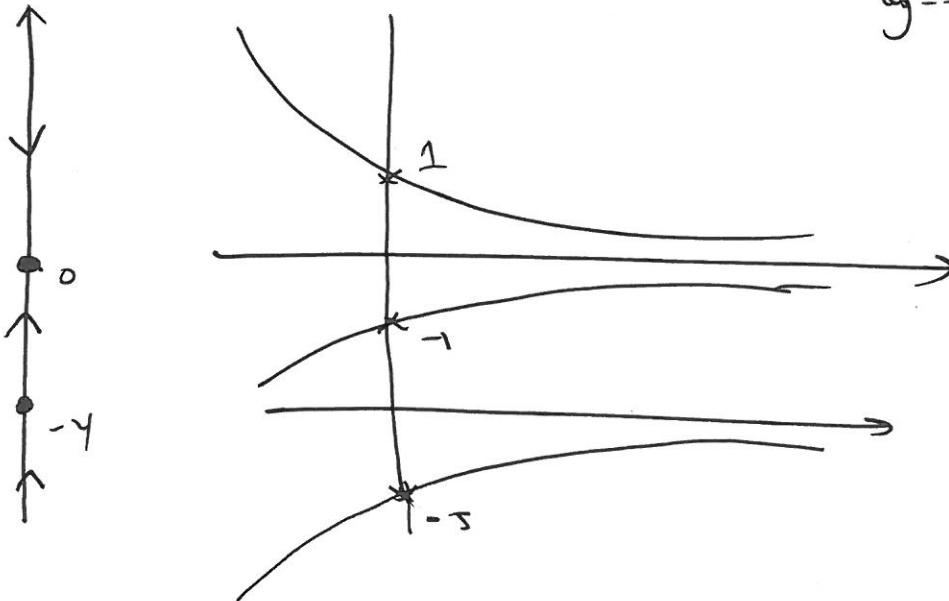


Draw the **phase line** of this differential equation. **Classify its equilibrium points.**

Draw the solutions satisfying $y(t_0) = -5$, $y(t_0) = -1$ and $y(t_0) = 1$.

two eq. solutions,

$$y = -4 \text{ and } y = 0$$



$y = 0$ is a sink (stable)

$y = -4$ is a node (semi-stable)

3. The air in a small rectangular room that is 20ft by 5ft by 10ft contains initially 40 liters of carbon monoxide. Air containing 0.1liters/ ft^3 is blown into the room at the rate of $100\text{ft}^3/\text{min}$ and well mixed air flows out through a vent at the same rate.

a. (8%) Write an initial value problem that describes the amount of carbon monoxide in the room. **Do not solve.**

let $Q(t)$ = amount of carbon monoxide in room $\Rightarrow Q(0) = \underline{40}$.

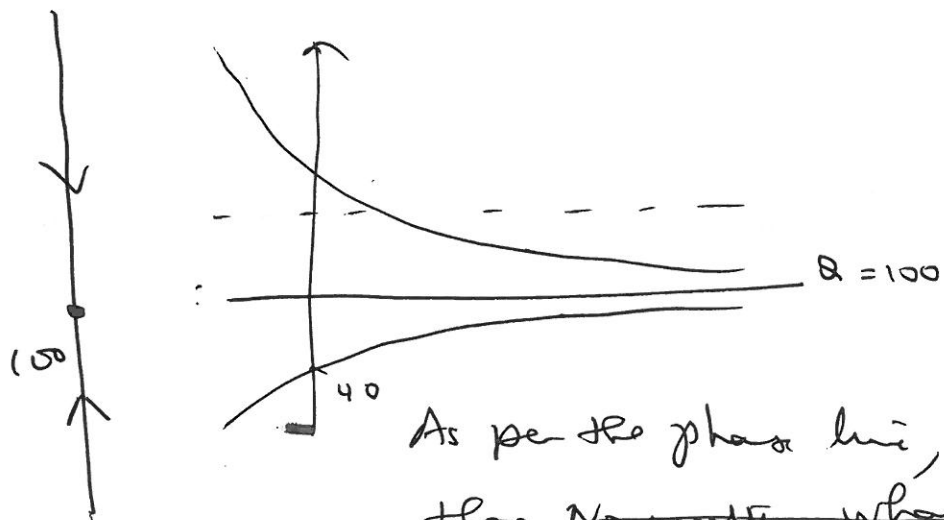
$$\text{input rate} = 0.1 \times 100 = 10 \text{ liters/min.}$$

$$\text{output rate} = \frac{Q(t)}{20 \times 5 \times 10} \times 100 = \frac{Q(t)}{10}$$

$$\therefore \frac{dQ}{dt} = 10 - \frac{Q(t)}{10}, \quad ; \quad Q(0) = 40.$$

- b. (5%) Sketch the phase line corresponding to this ODE and **conclude from the phase line** how much carbon monoxide will be in the room after a long period of time.

this is an aut. ODE : eq. 5 : $\frac{dQ}{dt} = 10 - \frac{Q}{10} \Rightarrow Q(t) = 100$
 $= 100$
 $= 100$



As per the phase line, starting from ~~the~~ ~~no matter~~ ~~what~~ ~~initial~~

5 with $Q(t) = 40$, solutions tend to $Q = 100$.

5. Consider the differential equation: $\frac{dy}{dt} = \alpha - y^2$

- a. (9%) Discuss the existence of equilibrium solutions for various values of α (that is for what values of α , are there two fixed points, one fixed point, or none).
eq. solutions eq. soln

$$\frac{dy}{dt} = 0 \Rightarrow \alpha - y^2 = 0 \Rightarrow y^2 = \alpha$$

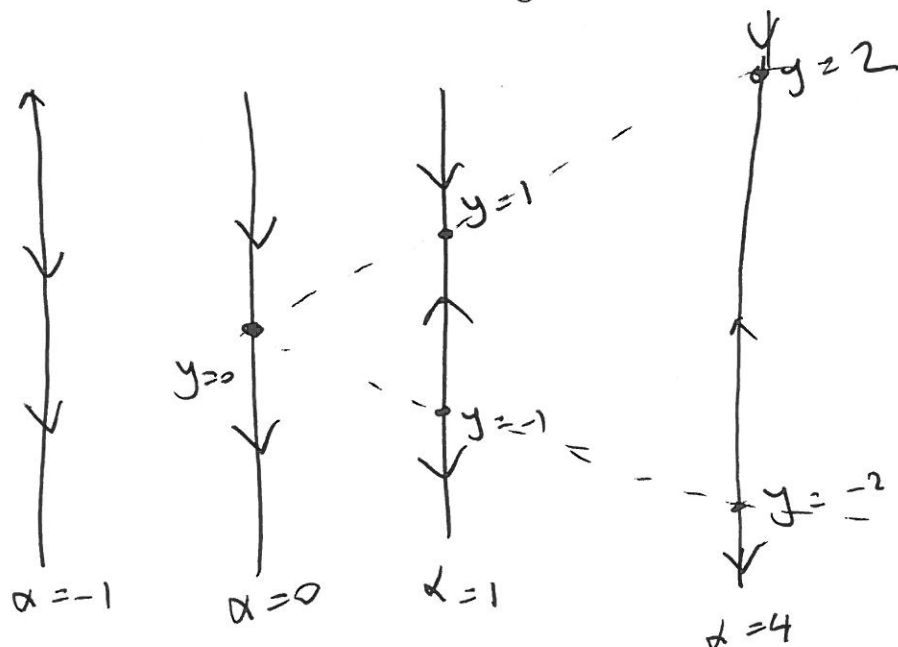
If $\alpha < 0$ \therefore no eq. solutions

If $\alpha = 0 \Rightarrow y = 0$ is the only eq. solution.

If $\alpha > 0$, then there are 2: $y = \pm \sqrt{\alpha}$.

- b. (9%) Draw the phase lines (all next to each other) for the the following cases: $\alpha = -1, \alpha = 0, \alpha = 1, \alpha = 4$ and classify the equilibrium points.

Bonus: Try to plot a graph that join the equilibrium points for the various values of α . This is called the **bifurcation diagram**.



6. (15%) Consider the 4 differential equations:

a. $\frac{dy}{dt} = (2-y)(y+1)$

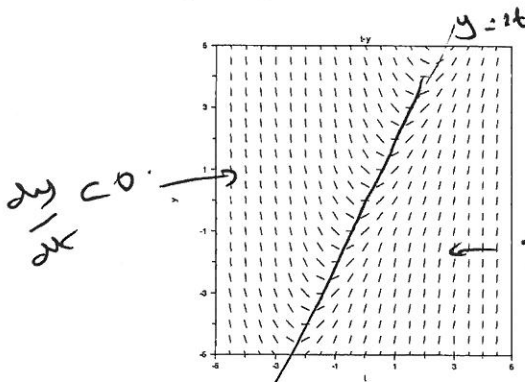
b. $\frac{dy}{dt} = t(y-1)$

c. $\frac{dy}{dt} = t^3 + 2$

d. $\frac{dy}{dt} = 2t - y$

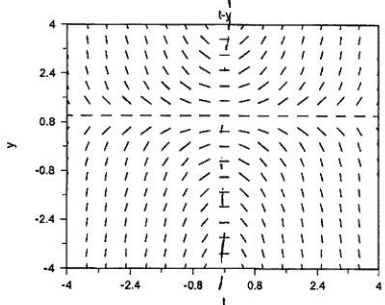
Match each direction field below with exactly one ODE and justify your answer.

I. The slope/direction field below matches d because:



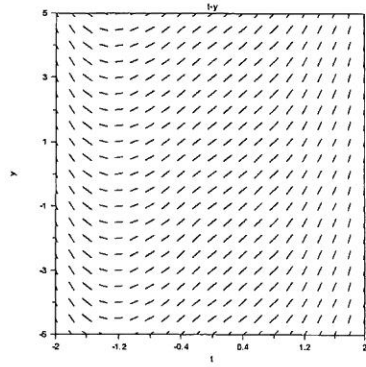
$y = 2t \Rightarrow \frac{dy}{dt} = 0 \checkmark$
 $y > 2t$ (above line), $\frac{dy}{dt} < 0 \checkmark$
 $y < 2t$, $\frac{dy}{dt} > 0 \checkmark$

II. The slope/direction field below matches b because:



Horizontal mini-tangents when $t = 0$
 or when $y = 1$.
 $t > 0, y > 1, \frac{dy}{dt} > 0$ | $t < 0, y > 1, \frac{dy}{dt} < 0$
 $t < 0, y < 1, \frac{dy}{dt} > 0$ | $t > 0, y < 1, \frac{dy}{dt} < 0$

III. The slope/direction field below matches c because:



All the mini-tangents are parallel to each other for a fixed value of t . ($t = \sqrt[3]{2}$).
 for $t > \sqrt[3]{2}$, $t^3 + 2 > 0 \Rightarrow$ mini tangents have positive slopes
 for $t < -\sqrt[3]{2}$, mini tangents have negative slopes.

Scratch