

LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS
MTH 304 – DIFFERENTIAL EQUATIONS
EXAM 1 – SPRING 2010

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ID#:

Instructions: This exam consists of 9 problems and a total of 9 pages. Make sure no problem/page is missing. Answer the questions in the space provided for each problem. If more space is needed, you can use the back of the pages. To receive full credits, you have to justify your answers.

Question Number	Grade
1. (14%)	
2. (10%)	
3. (8%)	
4. (18%)	
5. (10%)	
6. (12%)	
7. (15%)	
8. (8%)	
9. (5%)	
TOTAL	

1. (14%) Identify the following differential equations as separable, linear, or exact, then solve them analytically. The family of solutions may be given implicitly.

a. $y^2 \frac{dy}{dt} = e^{y^3 - 2t}$

b. $\frac{dy}{dt} + y \cot t = 2 \cos t$

a. $y^2 \frac{dy}{dt} = e^{y^3} \cdot e^{-2t} \Rightarrow \frac{y^2}{e^{y^3}} dy = e^{-2t} dt \text{ (separable)}$
 ~~$\Rightarrow \frac{1}{y^3} dy = e^{-2t} dt$~~ ~~$\Rightarrow \frac{1}{y^3} dy = -\frac{1}{2} e^{-2t} dt + C$~~ (implicit family)
 $\Rightarrow \frac{1}{y^2} e^{-y^3} = -\frac{1}{2} e^{-2t} + C$

b. $\frac{dy}{dt} + y \cot t = 2 \cos t \text{ (Linear)}$

Integrating factor: $\mu(t) = e^{\int \cot t dt} = e^{\ln |\sin t|} = |\sin t|$

$$\Rightarrow y(t) = \frac{1}{|\sin t|} \int |\sin t| 2 \cos t$$

$$= \frac{1}{\sin t} \int \sin(2t) dt$$

$$= \frac{1}{\sin t} \left[-\frac{1}{2} \cos(2t) + C \right].$$

2. (10%) Find the value of n for which the equation $(t + ye^{2ty})dt + nte^{2ty}dy = 0$ is exact and then solve the equation for that particular value of n .

$$M(t,y) = t + ye^{2ty} \quad ; \quad N(t,y) = nte^{2ty}$$

$$\text{To be exact, we need: } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

$$\Rightarrow e^{2ty} + 2tye^{2ty} = n[e^{2ty} + 2ye^{2ty}]$$

$$\Rightarrow \boxed{n = 1}$$

The family of solutions is: $\varphi(t,y) = C$

$$\text{where } \frac{\partial \varphi}{\partial t} = t + ye^{2ty} \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = te^{2ty}.$$

$$\text{Now, } \frac{\partial \varphi}{\partial y} = te^{2ty} \Rightarrow \varphi(t,y) = \frac{1}{2}e^{2ty} + f(t)$$

$$\text{Now, } \frac{\partial \varphi}{\partial t} = t + ye^{2ty} \Rightarrow \cancel{2y \cdot \frac{1}{2}e^{2ty}} + f'(t) = t + ye^{2ty}$$

$$\Rightarrow f'(t) = t \Rightarrow f(t) = \frac{t^2}{2}$$

\therefore Solutions are:

$$\frac{1}{2}e^{2ty} + \frac{t^2}{2} = C.$$

3. (8%) A brine solution of salt flows at a constant rate of 4L/min into a large tank that initially held 100L of pure water. The solution inside the tank is kept well stirred and flows out of the tank at the rate of 3L/min. If the concentration of salt in the brine entering the tank is 0.2 Kg/L, determine the amount of salt in the tank in t minutes. Also, determine the time when the concentration of salt reaches 0.1Kg/L.

Let $s(t)$ = amount of salt at time t ; $s(0) \neq 0$.

$$\frac{ds}{dt} = 4 \times 0.2 - \frac{s(t)}{100+t} \times 3 = 0.8 - \frac{3s}{100+t}$$

(see next page)

Linear equation: $\frac{ds}{dt} + \frac{3}{100+t} s = 0.8$

$$s(t) = e^{\int \frac{3}{100+t} dt} = e^{3 \ln(100+t)} = \frac{1}{(100+t)^3}$$

$$\Rightarrow s(t) = (100-t)^3 \int \frac{1}{(100-t)^3} \cdot 0.8 dt$$

$$= (100-t)^3 \left[\frac{1}{2(100-t)^2} + C \right] = \frac{100-t}{2} + C(100-t)^3$$

$$\text{But } s(0) = 0 \Rightarrow 0 = \frac{100}{2} + C(100)^3 \Rightarrow C(100)^3 = -\frac{100}{2}$$

$$\Rightarrow C = \frac{-1}{2(100)^2}$$

$$\frac{100-t}{2} + C(100-t)^3 = 0.1$$

$$\Rightarrow \frac{1}{2} + 0.1 = -C(100-t)^2 \Rightarrow t = \dots$$

4. Consider the separable differential equation $\frac{dy}{dt} = -3t^2y$ with initial condition $y(0) = 1$.

- a. (6%) Find the solution to this initial-value problem.

$$\frac{dy}{dt} = -3t^2y \Rightarrow \frac{dy}{y} = -3t^2 dt$$

$$\Rightarrow \ln|y| = -t^3 + C$$

$$\Rightarrow y = Ce^{-t^3}; \text{ But } y(0) = 1$$

$$\Rightarrow 1 = C \quad \boxed{y(t) = e^{-t^3}}$$

3. $S(t)$ = amount of salt at time t .

$$\frac{ds}{dt} = 0.8 - \frac{3s}{100+t} \Rightarrow \frac{ds}{dt} + \frac{3}{100+t} \frac{s}{\cancel{100+t}} = 0.8$$

$$\mu(t) = e^{\int \frac{3}{100+t} dt} = e^{3 \ln(100+t)} = e^{3 \ln(100+t)} = (100+t)^3$$

$$\Rightarrow s(t) = \frac{1}{(100+t)^3} \int (100+t)^3 \cdot 0.8 dt$$

$$\Rightarrow s(t) = \frac{1}{(100+t)^3} \left[\frac{(100+t)^4}{4} \cdot 0.8 + C \right]$$

$$\Rightarrow s(t) = 0.2(100+t) + \frac{C}{(100+t)^3}$$

$$\text{But } s(0) = 0 \Rightarrow 0 = 0.2 \times 100 + \frac{C}{100^3}$$

$$\Rightarrow 0 = 20 + \frac{C}{100^3} \Rightarrow C = -20(100)^3.$$

$$\therefore s(t) = 0.2(100+t) - \frac{20(100)^3}{(100+t)^3}$$

$$\text{Now, find } t / \frac{s(t)}{100+t} = 0.1 \Rightarrow \frac{0.2(100+t) - \frac{20(100)^3}{(100+t)^3}}{100+t} = 0.1$$

$$\Rightarrow 0.2 - \frac{20(100)^3}{(100+t)^4} = 0.1 \Rightarrow 0.1 = \frac{20(100)^3}{(100+t)^4}$$

$$\Rightarrow (100+t)^4 = 20(100)^4 \Rightarrow 100+t = \sqrt[4]{20}$$

$$\Rightarrow t = 100(\sqrt[4]{20} - 1)$$

b. (4%) Calculate the exact value of $y(0.9)$.

$$y(0.9) = e^{-6 \cdot 0.9^3} = \frac{1}{e^{\frac{9^3}{10^3}}} = \frac{1}{e^{27/100}}$$

c. (8%) Find an approximate value of $y(0.9)$ using Euler's method with a step size $\Delta t = 0.3$, then plot that approximation on the direction field below.

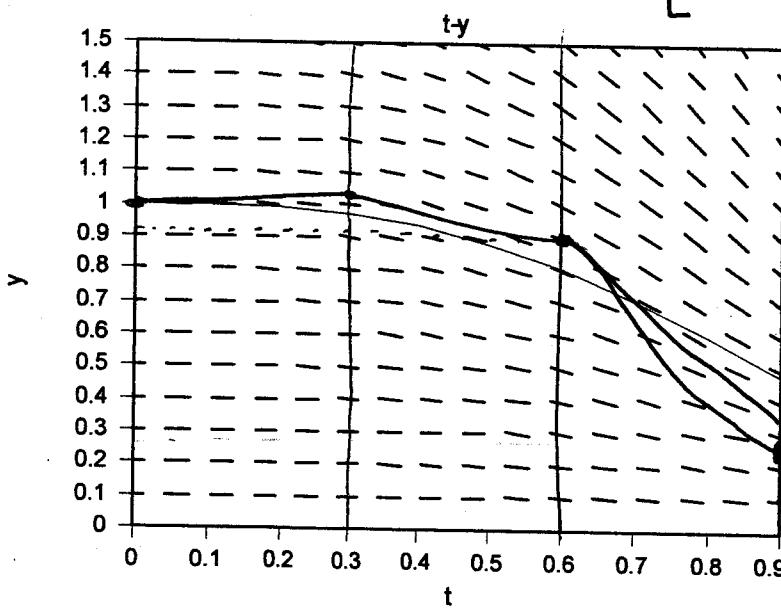
$$\boxed{y_0 = 1} ; \quad y_1 = \Delta t f(t_0, y_0) + y_0 = 0.3 f(0, 1) + 1 \\ \text{where } f(t, y) = -3t^2 y \Rightarrow f(0, 1) = 0. \Rightarrow \boxed{y_1 = 1}$$

$$y_2 = \Delta t f(t_1, y_1) + y_1 = 0.3 (-3 \times 0.3^2 \times 1) + 1 = \cancel{0.81} - \cancel{0.81} = \boxed{0.19}$$

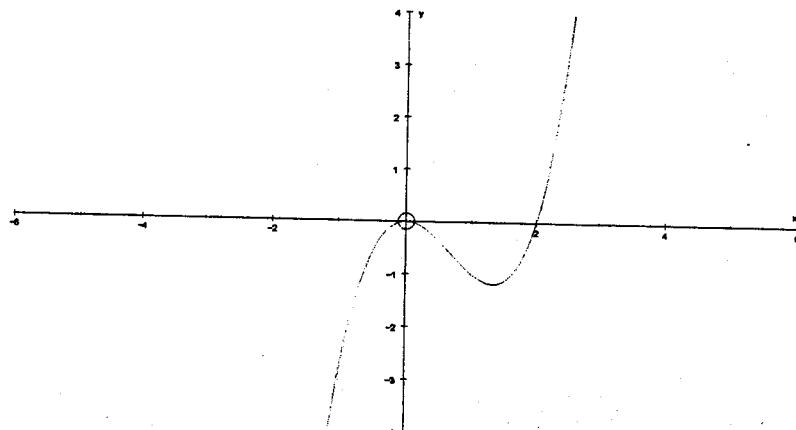
$$= -0.081 + 1 = 0.919.$$

$$y_3 = \Delta t f(t_2, y_2) + y_2 = 0.3 f(0.6, 0.919) + 0.919 \\ = 0.3 [-3 \times 0.6^2 \times 0.919] + 0.919$$

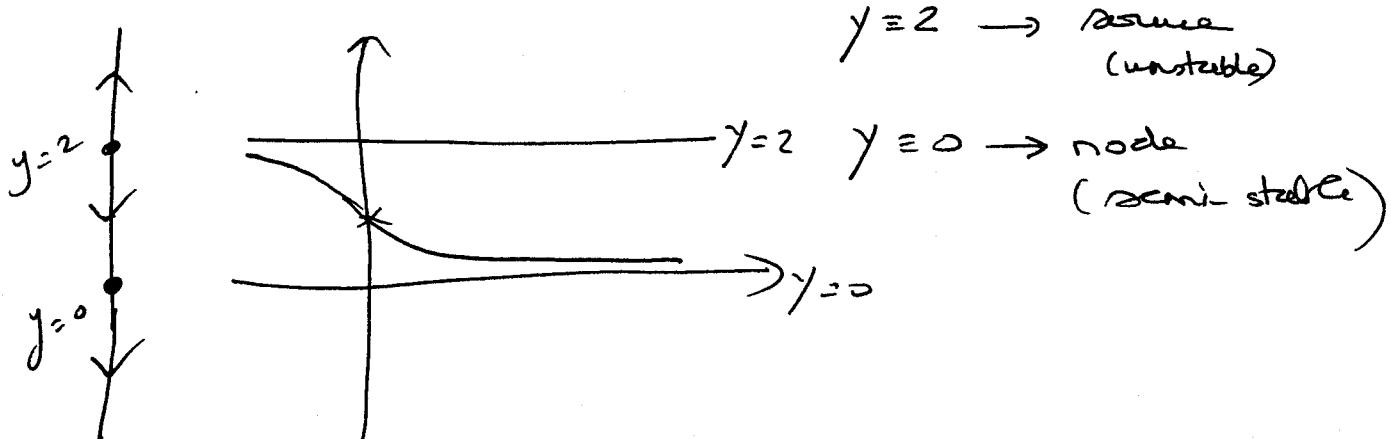
$$= 0.297756.$$



5. (10%) Consider the differential equation $\frac{dy}{dt} = f(y)$, where the graph of $f(y)$ takes the shape:



Draw the phase line of this autonomous equation; identify its equilibrium solutions; classify them and then discuss the shape of the solution satisfying the initial condition $y(0) = 1$.



Phase line Discussion of the solution satisfying $y(0) = 1$.
 Note that this solution is decreasing according to the phase line.

Now, as $y \rightarrow 2^-$, $f(y) \rightarrow 0^-$ $\therefore \frac{dy}{dt} \rightarrow 0^-$ — tangents are almost horizontal. For this reason, the solution is asymptotic to $y = 2$ (it does not cross $y = 2$ because $f(y)$ is continuous and differentiable \Rightarrow uniqueness applies).

The same argument works for as $y \rightarrow 0^+$.

6. Consider the family of ode's $\frac{dy}{dt} = y^2 - 4y + \alpha; \alpha \neq 0$

- (4%) Show that $\alpha = 4$ is a critical value for this problem in the sense that the number of equilibrium solutions depends on that value of α .
- (8%) Do the phase lines for this autonomous ode in the following cases: $\alpha = 0, \alpha = 2, \alpha = 4$, and $\alpha = 5$. In each case classify the equilibrium solution as stable/sink, unstable/source, or node/semi-stable.

(a) Equilibrium solns: $y^2 - 4y + \alpha = 0$
 $\Rightarrow y = 2 \pm \sqrt{4 - \alpha}$

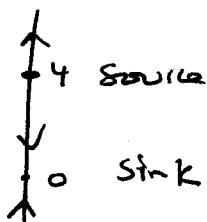
The number of eq. solutions depends on the sign of $4 - \alpha$.

If $4 - \alpha > 0 \Rightarrow \alpha < 4 \Rightarrow 2$ eq. solns

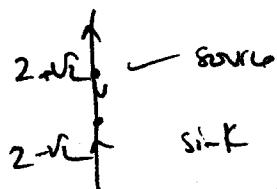
If $4 - \alpha = 0 \Rightarrow \alpha = 4 \Rightarrow 1$ eq. solution

If $4 - \alpha < 0 \Rightarrow \alpha > 4 \Rightarrow$ no eq. solutions.

(b) $\alpha = 0 \Rightarrow$ 2 eq. solutions $y^2 - 4y = 0 \Rightarrow y(y-4)$ $\begin{cases} y=0 \\ y=4 \end{cases}$



$\alpha = 2 \Rightarrow y^2 - 4y + 2 = 0 \Rightarrow 2$ solutions $\begin{cases} y = 2 + \sqrt{2} \\ y = 2 - \sqrt{2} \end{cases}$



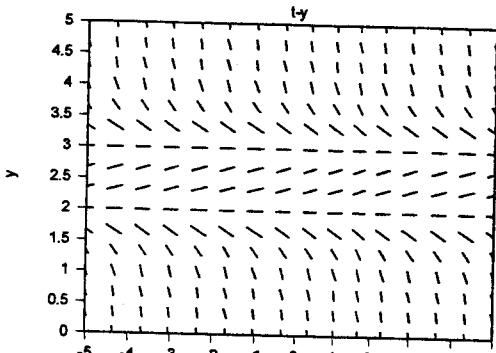
$\alpha = 4 \Rightarrow 1$ eq. soln $\begin{cases} 2 \text{ node} \end{cases}$

$\alpha = 5 \Rightarrow$ no eq. solutions

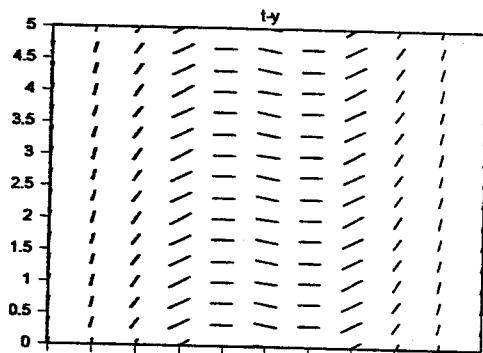
7. (15%) Consider the following first-order odes

$$\frac{dy}{dt} = (2-y)(3-y); \quad \frac{dy}{dt} = (y-2)(3-y); \quad \frac{dy}{dt} = (2-t)(3+t); \quad \frac{dy}{dt} = (2-y)(3+t),$$

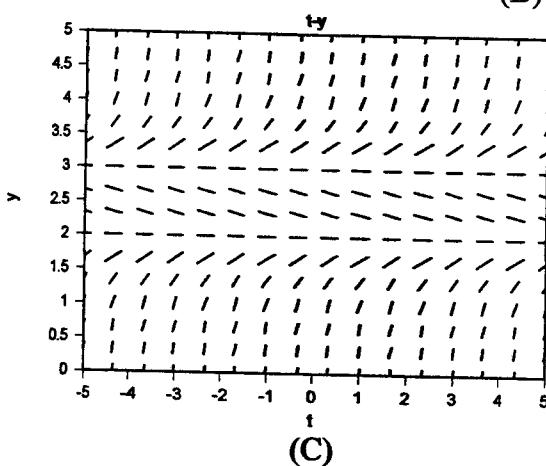
Assign the direction fields below to the corresponding ode and write a short paragraph to justify your choices.



(A)



(B)



(C)

We can disregard $\frac{dy}{dt} = (2-y)(3+t)$ since in (A) and (C) solutions

(i) are horizontal translates and in (B) solutions are vertical translates.

(ii) since in (B), solutions are vertical translates \Rightarrow the corresponding DE is $\frac{dy}{dt} = (2-t)(3+t)$

for $\frac{dy}{dt} = (2-y)(3-y)$

$$\begin{array}{c} \uparrow y=3 \\ \downarrow y=2 \\ \vdash \end{array} \rightarrow (C)$$

$\frac{dy}{dt} = (y-2)(3-y)$

$$\begin{array}{c} \uparrow y=3 \\ \downarrow y=2 \\ \vdash \end{array} \rightarrow (A)$$

8. Consider the IVP $\frac{dy}{dt} = \frac{y}{t^2}; y(1) = \frac{1}{e}$

a. (2%) Show that the function $f(t) = e^{-1/t}; t > 0$ satisfies this IVP

$$\frac{dy}{dt} = \frac{1}{t^2} \cdot e^{-1/t} = \frac{1}{t^2} \cdot y \quad \text{and} \quad y(1) = e^{-1} = \frac{1}{e} \quad \checkmark$$

b. (2%) Show also that the function $g(t) = \begin{cases} 0; t \leq 0 \\ e^{-1/t}; t > 0 \end{cases}$ satisfies the same ode.

This question is wrong
because $t \neq 0$.

c. (4%) Explain why this does not contradict the uniqueness theorem.

9. (5%) Imagine yourself standing in front of an audience with minimal calculus background. Your task is to introduce to your audience in the simplest way the concept of a (ordinary) differential equation. Elaborate in a short paragraph how you would complete this task. Support your ideas with examples and describe/explain the various approaches to solve such equations.