Lebanese American University

Byblos

Diff. Equations
Summer II, 2011
Test # 1
90 minutes

Name:

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1. The temperature of an object placed in a room with constant temperature of $25^\circ C$, decreased from $75^\circ C$ to $65^\circ C$ in two minutes. Find the temperature T(t), (10 pts)

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2. Find an integrating factor of the form $\rho = x^n y^m$, then solve the first-order DE: $(2y^4 + 3xy)\,dx + (2xy^3 - x^2)\,dy = 0$ (15 pts)

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3. Solve the Bernoulli DE: $xy' + 4y = 2x\sqrt{y}\,\sin x$ (10 pts)

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4. Solve the second order Cauchy-Euler equation
$$2x^2 y'' + 3xy' - y = x - \frac{1}{x^2}$$ (15 pts)

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5. Solve the second-order LDE: $y'' - 9y = 2x + xe^{3x}$ (12 pts)

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6. a) A mass of 2 kg stretches a spring 0.2 meter.
Find the spring constant k.
b) We push the mass 0.2 meter upward and release it with an initial upward velocity of 5m/sec. Find the position of m. (10 pts)

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7. Find a LDE whose solutions are $y_1 = \frac{\cos(3\,\mathrm{Ln}\,x)}{x^2}$ and $y_2 = \frac{\sin(3\,\mathrm{Ln}\,x)}{x^2}$ (10 pts)

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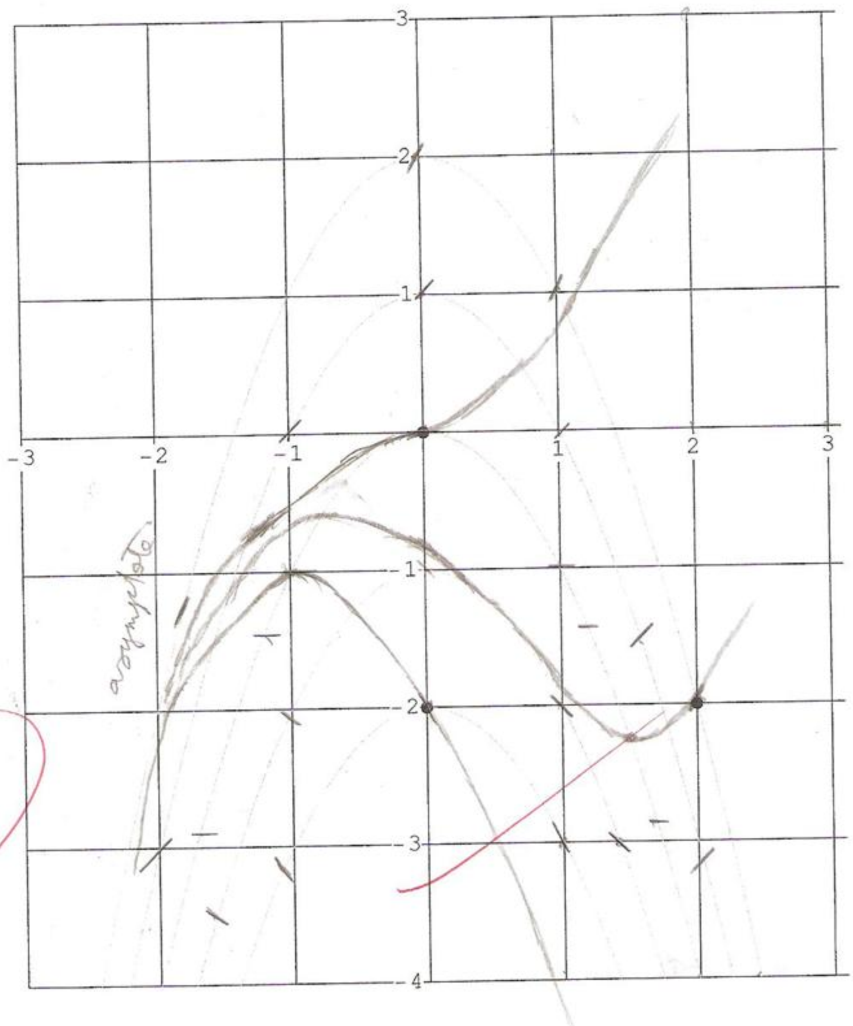
8. Solve the second-order LDE: $xy'' + 2y' - xy = 0$.
Given: $y_1 = \frac{e^x}{x}$ is a solution. (8 pts)

Turn over

9. Draw the direction field of the DE: $y' - y = x^2$
and sketch three solutions passing respectively through
the points : $(0, -2)$, $(0, 0)$ and $(2, -2)$. (10 pts)



$m=0 \rightarrow y=-x^2$

$m=1 \rightarrow y=-x^2+1$

$m=2 \rightarrow y=-x^2+2$

$m=-1 \rightarrow y=-x^2-1$

$m=-2 \rightarrow y=-x^2-2$

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EXAMINATION BOOKLET

Name: Charbel Fakhry

Subject: MTH 304.

Instructor:

98/100

(1)

Date:

I.D. No. 201001730.

Section:

Box No.

10, 25, 35, 50, 62, 72, 81, 89,

(Begin here and write on both sides)

1) $T_M = 25$

$T(0) = 75$

$T(2) = 65$

$T' = -k(T - T_M)$

$$\frac{dT}{dt} = -k(T-25)$$

$$\frac{dT}{T-25} = -k\,dt$$

~~$\ln |T-25|$~~

$$\int \frac{dT}{T-25} = \int -k\,dt$$

seperable.

$$\ln \frac{T-25}{C} = -kt$$

$$\frac{T-25}{C} = e^{-kt}$$

$$\boxed{T = C \cdot e^{-kt} + 25}$$

$T(0) = 75 \Rightarrow 75 = c e^{0k} + 25$

$$c = 75 - 25 = 50$$

$T(2) = 65 \Rightarrow$

$65 = 50 e^{-2k} + 25$

$50 e^{-2k} = 40$

$e^{-2k} = \frac{4}{5} \Rightarrow$ $e^{-k} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

(10)

$$T = 50 \cdot \left(\frac{2\sqrt{5}}{5}\right)^t + 25$$

2] Exactness: $\underbrace{(2y^4 + 3xy)}_{P} dx + \underbrace{(2xy^3 - x^2)}_{Q} dy = 0$

$P_y = 8y^3 + 3x$

$Q_x = 2y^3 - 2x$

$P_y \neq Q_x$ then ω is not exact.

let $\rho = \rho(x,y)$ such that $\rho\omega$ is exact.

$$\underbrace{\rho(2y^4 + 3xy)}_{P} dx + \underbrace{\rho(2xy^3 - x^2)}_{Q} dy = 0$$ (1)

$\Rightarrow P_y = Q_x$

$\rho_y(2y^4 + 3xy) + \rho(8y^3 + 3x) = \rho_x(2xy^3 - x^2) + \rho(2y^3 - 2x)$

II)

$$\rho_y(2y^4+3xy)+\rho(6y^3+5x)=\rho_x(2xy^3-x^2) \quad (3)$$

Let $\rho = x^n y^m$

$\Rightarrow \rho_x = n.x^{n-1}.y^m$

$\Rightarrow \rho_y = m.y^{m-1}.x^n$

Replace in (3).

$$m.y^{m-1}.x^n(2y^4+3xy)+x^ny^m(6y^3+5x)-nx^{n-1}y^m(2xy^3-x^2)=0$$

$$m.y^m x^n(2y^3+3x)+x^ny^m(6y^3+5x)-nx^ny^m(2y^3-x)=0$$

$$m(2y^3+3x)+6y^3+5x-n(2y^3-x)=0$$

$$2my^3+3mx+6y^3+5x-2ny^3+nx=0$$

$$x(3m+5+n)+y^3(2m+6-2n)=0$$

$$\begin{cases} ~~3m+n=5~~ \; 3m+n+5=0 \\ 2m+6-2n=0 \end{cases} \Rightarrow \begin{matrix} m=-2 \\ n=1. \end{matrix}$$

$$\Rightarrow \rho = x^ny^m = x.y^{-2} = \frac{x}{y^2}$$

$$\boxed{\rho = \frac{x}{y^2}}$$

Replace in (2).

$$\frac{x}{y^2}\left(2y^4+3xy\right)dx + (2xy^3-x^2)\,dy = 0.$$

$$\underbrace{\left(2xy^2+\frac{3x^2}{y}\right)}_{P}dx + \underbrace{\left(2x^2y-\frac{x^3}{y^2}\right)}_{Q}dy = 0.$$

ω is exact $\omega = df$

thus $f_x = P$ and $f_y = Q$

$$\Rightarrow f_x = 2xy^2 + \frac{3x^2}{y} \Rightarrow f = \int\left(2xy^2+\frac{3x^2}{y}\right)dx$$

$$\boxed{f = x^2y^2 + \frac{x^3}{y} + g(y)}$$

$$f_y = 2yx^2 - \frac{x^3}{y^2} + g'(y)$$

but $f_y = Q \Rightarrow g'(y) = 0$

$$\Rightarrow \boxed{g(y) = c} \quad \text{take} \quad \boxed{c=0}$$

$$\Rightarrow \boxed{f = x^2y^2 + \frac{x^3}{y}}$$

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Now, $df = \omega = 0 \Rightarrow f = c.$

gen sol^o: $\boxed{f = c = x^2y^2 + \frac{x^3}{y}}$

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(2)

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3) $xy' + 4y = 2x\sqrt{y}\sin x$.

$y' + \frac{4}{x}y = 2\sin x \; y^{1/2}$.

$y^{-1/2}y' + \frac{4}{x}y^{1/2} = 2\sin x$.

let $z = y^{1/2}$ $\Rightarrow z' = \frac{1}{2}y'y^{-1/2}$

$\Rightarrow y'y^{-1/2} = 2z'$

$2z' + \frac{4}{x}z = 2\sin x$.

$z' + \frac{2}{x}z = \sin x$. (LDE).

$a(x) = \frac{2}{x}$

$r(x) = \sin x$.

$z = \frac{1}{\rho}\int \rho r \, dx$ where $\rho = e^{\int a(x)\,dx}$

$\rho = e^{\int \frac{2}{x}dx}$

$\rho = e^{2\ln|x| + \cancel{k}^{\to 0}}$

$z = \frac{1}{x^2}\int x^2 \sin x \, dx$.

$\rho = e^{\ln x^2}$

$\rho = x^2$

Tabular:

derivative		integral
x^2	$+$	$\sin x$
$2x$	$-$	$-\cos x$
2	$+$	$-\sin x$
0		$\cos x$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

so $$y = \frac{1}{x^2}\left(-x^2 \cos x + 2x \sin x + 2\cos x + c\right)$$

$$y = -\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2} + \frac{c}{x^2}$$

10 $\Rightarrow$ $$\boxed{y = -\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2} + \frac{c}{x^2}}$$

4) $2x^2y'' + 3xy' - y = x - \frac{1}{x^2}$

~~Hom~~ $\Rightarrow$ $x^2y'' + \frac{3}{2}xy' - \frac{1}{2}y = \frac{1}{2}x - \frac{1}{2x^2}$

Homogeneous sol$^{\circ}$:

$x^2y'' + \frac{3}{2}xy' - \frac{1}{2}y = 0.$

let $|x| = e^t \Rightarrow t = \ln|x|$.

$\Rightarrow \ddot{y} + \frac{1}{2}\dot{y} - \frac{1}{2}y = 0$

$\lambda^2 + \frac{1}{2}\lambda - \frac{1}{2} = 0$

$\Delta = \frac{9}{4} > 0$

$\Rightarrow \lambda_1 = \frac{-b + \sqrt{\Delta}}{2} = \frac{1}{2}$

$\lambda_2 = \frac{-b - \sqrt{\Delta}}{2} = -1$

$y_h = k_1 e^{(1/2)t} + k_2 e^{-t}$

$\Rightarrow$ $y_h = k_1\sqrt{x} + k_2 |x|^{-1}$

Particular sol^t°: Sum rule: $(y_p = y_{p_1} + y_{p_2})$.

$$x^2y'' + \frac{3}{2}xy' - \frac{1}{2}y = \frac{1}{2}x. \quad (3)$$ Cauchy method.

(3) has a particular solution of the form

$$y = ax \Rightarrow y' = a \Rightarrow y'' = 0$$

replace in (3)

$$\Rightarrow 0 + \frac{3}{2}ax - \frac{1}{2}ax = \frac{1}{2}x.$$

$$\frac{3}{2}a - \frac{1}{2}a = \frac{1}{2}.$$

$$\boxed{a = \frac{1}{2}} \Rightarrow \boxed{y_{p_1} = \frac{1}{2}x}$$

y_{p_2}: $$x^2y'' + \frac{3}{2}xy' - \frac{1}{2}y = -\frac{1}{2}x^{-2} \quad (4)$$ "Cauchy method"

(4) has a part. sol^t° of the form $y = bx^{-2} \Rightarrow y' = -2bx^{-3}$

$$\Rightarrow y'' = 6bx^{-4}$$

replace in (4):

$$x^2.(6bx^{-4}) + \frac{3}{2}x.(-2bx^{-3}) - \frac{1}{2}bx^{-2} = -\frac{1}{2}x^{-2}.$$

$$6b - 3b - \frac{1}{2}b = -\frac{1}{2} \Rightarrow \frac{5}{2}b = -\frac{1}{2} \quad \boxed{b = -\frac{1}{5}}$$

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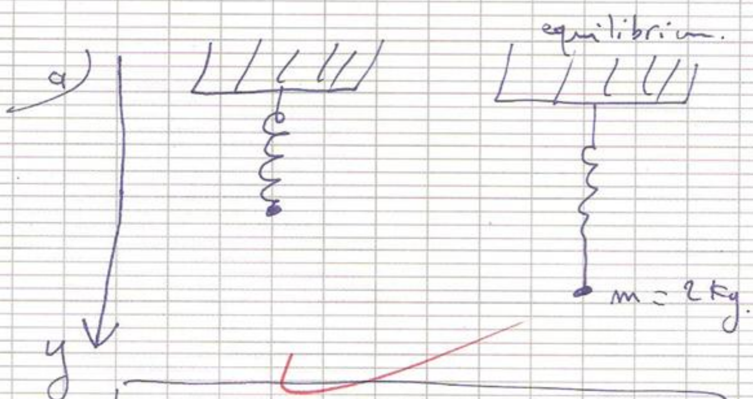
$$y_{p_2} = -\frac{1}{5} x^{-2}$$

(15) gen sol^{n}:

$$y = y_h + y_p$$

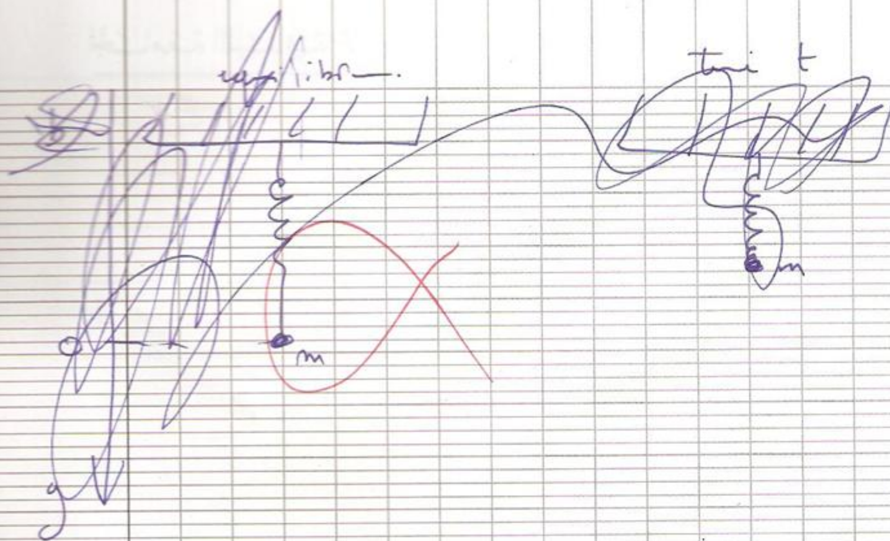
$$y = k_1 \sqrt{x} + k_2 |x|^{-1} + \frac{1}{2} x - \frac{1}{5} x^{-2}$$

(6) a)



$$k = \frac{m \cdot g}{s_0} = \frac{2 \times 9,8}{0,2} = 98$$

b)



b) $y(0) = -0.2$

$y'(0) = -5$ (upward velocity)

$\Rightarrow$ Hooke's law

$F = -k\Delta L = -ky = -98y$

Newton's law:

$F = ma = my'' = 2y''$

$F = F$

$2y'' = -98y$

$$\boxed{y'' + 49y = 0} \qquad \boxed{\omega^2 = 49}$$

HLDEC.

$\Rightarrow \lambda^2 + 49 = 0 \Rightarrow \lambda^2 = 49 i^2$

$\Rightarrow \lambda_1 = 7i$

$\lambda_2 = -7i$

$\begin{cases} \alpha = 0 \\ \beta = 7 \end{cases}$

$\Rightarrow$ $y = k_1 \cos 7t + k_2 \sin 7t$.

$y(0) = -0.2$.

$-0.2 = k_1$

* $y' = -7k_1 \sin 7t + 7k_2 \cos 7t$

$y'(0) = -5$

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$\Rightarrow -5 = 7k_2 \Rightarrow k_2 = \frac{-5}{7}$.

$y = -0.2 \cos 7t - \frac{5}{7} \sin 7t$.

7 $y_1 = \frac{\cos(3\ln x)}{x^2}$ $\quad y_2 = \frac{\sin(3\ln x)}{x^2}$

$y_1 = x^{-2} \cos(3\ln x)$ $\quad y_2 = x^{-2} \sin(3\ln x)$.

let $|x| = e^t \Rightarrow (x > 0)$ so $x = e^t$ and $\ln x = t$.

$y_1 = e^{-2t} \cos 3t$ $\quad y_2 = e^{-2t} \sin 3t$. $\quad \begin{pmatrix} \alpha = -2 \\ \beta = 3 \end{pmatrix}$.

y_1 and y_2 are 2 solutions to the HLDEC

whose characteristic equation has 2 complex roots.

$\lambda_1 = -2 + 3i$ $\quad \lambda_2 = -2 - 3i$.

~~so (E)~~ $\longrightarrow$

So, $(\lambda+2+3i)(\lambda+2-3i)$

$= (\lambda+2)^2 - 3i^2$

$= \lambda^2 + 4 + 4\lambda + 3$ 9

$= \lambda^2 + 4\lambda + 7$

So $\ddot{y} + 4\dot{y} + 7y = 0$ HLDEC.

$a - 1 = 4 \Rightarrow a = 5$

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So Euler cauchy:

DE: $x^2 y'' + 5x y' + 7y = 0$

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(8) $xy'' + 2y' - xy = 0$

$\Rightarrow y'' + \frac{2}{x} y' - y = 0.$

$y_1 = \frac{e^x}{x} \Rightarrow a(x) = \frac{2}{x}.$

$y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int a(x)\,dx}\,dx$

$e^{-\int \frac{2}{x}dx} = e^{-2\ln\frac{x}{c}} = e^{2\ln\frac{c}{x}} = e^{\ln\frac{c^2}{x^2}} = \frac{c^2}{x^2}$

take $c=1$ $\Rightarrow$

$\rightarrow$ $e^{-\int \frac{2}{x}dx} = \frac{1}{x^2}$

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$$y_2 = \frac{e^x}{x}\int \frac{x^2}{e^{2x}} \cdot \frac{1}{x^2}\, dx$$

$$y_2 = \frac{e^x}{x}\int e^{-2x}\, dx$$

$$y_2 = \frac{e^x}{x} \cdot \left(-\frac{1}{2} e^{-2x} + c^{\to 0}\right)$$

$$y_2 = -\frac{1}{2}\frac{e^{-x}}{x}$$

gn soln: $y = c_1 y_1 + c_2 y_2$

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$$\Rightarrow y = c_1 \frac{e^x}{x} + c_2 \cdot \left(-\frac{1}{2}\right) \cdot \frac{e^{-x}}{x}$$

(5) $y'' - 9y = 2x + xe^{3x}$

Homo. sol^{n}:

$y'' - 9y = 0$

$\lambda^2 - 9 = 0$

$\lambda_1 = 3 \qquad \lambda_2 = -3.$

$\Rightarrow$ ~~$y_h = k_1 e^{3t} + k_2 e^{-3t}$~~ ~~$y_1 = e^3$~~

$y_h = k_1 e^{3x} + k_2 e^{-3x}$

part sol^{n}: Sum rule.

$y'' - 9y = 2x$ (3). Undetermined coeff.

$r(x) = 2x = P_1(x) \rightarrow$ (3) has a particular sol^{n} of the form.

$y = x^m g(x)$

$y = x^m . (ax+b)$.

where $m=0$ because $x, 1$ are not hom. sol^{n}.

$y = ax+b \rightarrow y' = a \Rightarrow y'' = 0$

replace in (3) $\longrightarrow$ ~~$0x$~~ $- 9ax - 9b = 2x$

$\begin{cases} -9a = 2 \\ -9b = 0 \end{cases}$ $\boxed{a = -\frac{2}{9}}$ $\rightarrow \boxed{b = 0}$

$$y_{P_1} = -\frac{2}{9}x$$

y_{P_2}: $\quad y'' - 9y = xe^{3x} \quad (4) \quad$ Und. coeff.

$r(x) = xe^{3x} = P_1(x) \cdot e^{3x}$

then (4) has a part sol^{n} of the form:

$$y = x^m \cdot (ax+b) \cdot e^{3x}$$

where $m \neq 0$ because e^{3x} ~~part~~ hom sol^{n}
$m=1$ because x^2e^{3x}, xe^{3x} not hom. sol^{n}

$\rightarrow$ $y = (ax^2+bx)e^{3x}$.

$y' = 3e^{3x}(ax^2+bx) + e^{3x}(2ax+b)$

$$y' = e^{3x}(3ax^2 + 3bx + 2ax + b)$$

$y'' = 3e^{3x}(3ax^2 + 3bx + 2ax + b) + e^{3x}(6ax + 3b + 2a)$.

$y'' = e^{3x}(9ax^2 + 9bx + 6ax + 3b + 6ax + 3b + 2a)$.

$$y'' = e^{3x}(9ax^2 + 12ax + 9bx + 2a + 6b).$$

Replace in (4).

$$e^{3x}\left(9ax^2+12ax+9bx+2a+6b\right)-9\left(e^{3x}\left(ax^2+bx\right)\right)=xe^{3x}$$

$$e^{3x}\left(9ax^2+12ax+9bx+2a+6b-9ax^2-9bx\right)=xe^{3x}$$

$$12a\,xe^{3x}+e^{3x}(2a+6b)=xe^{3x}$$

$$\begin{cases}12a=1\\2a+6b=0\end{cases}$$

$$a=\frac{1}{12}\qquad b=-\frac{1}{36}$$

$$y_{p_2}=\left(\frac{1}{12}x^2-\frac{1}{36}x\right)e^{3x}$$

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gn sol^{o}:

$$y=k_1e^{3x}+k_2e^{-3x}-\frac{2}{9}x+e^{3x}\left(\frac{1}{12}x^2-\frac{1}{36}x\right).$$

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$$y'\geqslant 0$$

$$x^2+y\geqslant 0$$

$$y\geqslant -x^2$$ (increasing above $y=-x^2$).

α 1

$$y''=y+2x=y+x^2+2x$$