Lebanese American University
Department of Computer Science and Mathematics
MTH304 – Differential Equations
Exam 2 – Spring 2012

Time: 75 Minutes

Name: KEY **ID#:**

Instructions: This exam consists of 6 problems and a total of 8 pages (last page is for scratch). Make sure no problem/page is missing. Answer the questions in the space provided for each problem. If more space is needed, you can use the back of the pages. To receive full credits, you have to justify your answers.

Question Number	Grade
1. 24%	
2. 10%	
3. 15%	
4. 15%	
5. 24%	
6. 12%	
TOTAL	

1. (24%) Find the general solutions of the following second-order differential equations:

a. $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} = -8\cos(2t)$

$r^2 - 3r = 0 \Rightarrow r = 0$ or $r = 3$

$\therefore y_1 = k_1;\ y_2 = e^{3t}$

$y_p = A\sin(2t) + B\cos(2t)$

$y_p' = 2A\cos(2t) - 2B\sin(2t)$

$y_p'' = -4A\sin(2t) - 4B\cos(2t)$

Substitute: $-4A\sin(2t) - 4B\cos(2t) - 6A\cos(2t) + 6B\sin(2t)$

$= -8\cos(2t)$

$\Rightarrow \sin(2t)[-4A + 6B] + \cos(2t)[-4B - 6A] = -8\cos(2t)$

$\therefore -4A + 6B = 0 \rightarrow \boxed{A = \frac{3}{2}B}$

$-4B - 6A = -8$

$\Rightarrow 2B + 3A = 4$

$2B + \frac{9}{2}B = 4 \Rightarrow \frac{13B}{2} = 4$

$\boxed{B = \frac{8}{13}}$

and $\boxed{A = \frac{12}{13}}$.

b. $\frac{d^2y}{dt^2} - 9y = 9t^2 - e^{-3t}$

$r^2 - 9 = 0 \Rightarrow r = \pm 3 \Rightarrow y_1 = e^{3t};\ y_2 = e^{-3t}$

$y_p = at^2 + bt + c + Kte^{-3t}$

$y_p' = 2at + b + Ke^{-3t} - 3Kte^{-3t}$

$y_p'' = 2a - \underline{3Ke^{-3t}} - \underline{3Ke^{-3t}} + 9Kte^{-3t}$

Substitute:

$2a - \underline{6Ke^{-3t}} + \cancel{9Kte^{-3t}} - 9at^2 - 9bt - 9c - \cancel{9tK}e^{-3t} = 9t^2 - e^{-3t}$

$\therefore\ -6K = -1 \Rightarrow \boxed{K = 1/6}$

$-9a = 9 \Rightarrow \boxed{a = -1}$.

$-9b = 0 \Rightarrow b = 0$

$2a - 9c = 0 \Rightarrow c = \frac{2}{9}a \Rightarrow \boxed{c = -\frac{2}{9}}$.

2. (10%) Find <u>only</u> the general form of a particular solution to the following second-order differential equation:

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=3te^{-2t}$$

$r^2+4r+4=0 \Rightarrow r=-2$ is a double root $\therefore y_1=e^{-2t};\ y_2=te^{-2t}$

(1) $y_p = (at+b)e^{-2t} = ate^{-2t}+be^{-2t}$ ✗

(2) $y_p = t(at+b)e^{-2t} = at^2e^{-2t}+bte^{-2t}$ ✗

(3) $y_p = t^2(at+b)e^{-2t}$ ✓

3. (15%) Find the general solution to the following non-homogeneous equation

$$\frac{d^2y}{dt^2}+y=\tan t$$

$r^2+1=0 \Rightarrow r=\pm i \therefore y_1=\cos t;\ y_2=\sin t.$

A particular solution is: $y_p = v_1y_1+v_2y_2$, where

$\boxed{v_1 = -\int \frac{y_2\cdot\tan t}{W[y_1,y_2]}dt}$; $\boxed{v_2 = \int \frac{y_1\tan t}{W[y_1,y_2]}dt}$

Now $\boxed{W[y_1,y_2]} = y_1y_2' - y_2y_1' = \cos t(\cos t) - \sin t(-\sin t) = \boxed{1}$

$\therefore v_1 = -\int \sin t\times\tan t\,dt = -\int \frac{\sin^2 t}{\cos t}dt = -\int \frac{1-\cos^2 t}{\cos t}dt$

$= -\int(\sec t - \cos t) = \boxed{-\ln|\sec t+\tan t| + \sin t} = v_1$

$v_2 = \int \cos t\,\tan t\,dt = \int \sin t\,dt = \boxed{-\cos t} = v_2$

4. (15%) Consider the following homogeneous equation with variable coefficients:

$$t^2\frac{d^2y}{dt^2}-6y=0;$$

Given that $f(t)=t^3$ is one solution to this equation, find its general solution assuming that $t>0$.

We need y_p; $y_p = t^3 v(t)$; $v(t)$ to be determined.

$y_p' = 3t^2v + t^3v'$; $y_p'' = 6tv + 3t^2v' + 3t^2v' + t^3v''$

Substitute: $\cancel{t^2(6tv + 3t^2v' + 3t^2v + t^3v') - 6t^3v = 0}$

$\cancel{6t^3v + 3t^4v' + 3t^4v + t^5v' - 6t^3v = 0}$

$\therefore \cancel{3v' + 3v + tv' = 0}$

$\cancel{(3+t)v' = -3v \Rightarrow \frac{v'}{v} = \frac{-3}{3+t}}$ $\cancel{\therefore \ln|v| = -3\ln|3+t|}$, $t>0$

$\therefore v = \cancel{\frac{1}{(t+3)^3}}$.

$\therefore$ ~~A second solution is~~: $\cancel{\frac{t^3}{(t+3)^3}}$.

Substitute: $t^2(6tv + 6t^2v' + t^3v'') - 6(t^3v) = 0$

$\cancel{6t^3}v + 6t^4v' + t^5v'' - \cancel{6t^3}v = 0$

$\Rightarrow \frac{v''}{v'} = -\frac{6}{t} \Rightarrow v' = -6\ln|t| = \frac{1}{t^6}$ $\therefore v = \int t^{-6}dt = \frac{-1}{5t^5}$.

$\therefore$ A second solution is: $t^3\left(-\frac{1}{5t^5}\right) = \boxed{\frac{-1}{5t^2}}$

5. Consider the third-order homogeneous linear differential equation:

$$\frac{d^3y}{dt^3}+3\frac{d^2y}{dt^2}-4y=0;$$

a. (10%) Find three solutions to this homogenous equation.

$r^3+3r^2-4=0 \longrightarrow r=\pm1, \pm2, \pm4$

$r=1 \longrightarrow 1+3-4=0$

$$\begin{array}{r|l} r^3+3r^2-4 & r-1 \\ \hline -r^3+r^2 & r^2+4r+4 \\ 4r^2-4 & \\ -4r^2+4r & \\ 4r-4 & \\ -4r+4 & \\ 0 & \end{array}$$

$\therefore r^3+3r-4 = (r-1)(r^2+4r+4) = (r-1)(r+2)^2$

$\therefore y_1 = e^t ; \; y_2 = e^{-2t} ; \; y_3 = te^{-2t}$

b. (14%) Show that these solutions are independent and then solve the initial-value problem with the following initial conditions: $y(0)=0; y'(0)=2; y''(0)=8$.

$$W[y_1, y_2, y_3] = \begin{vmatrix} e^t & e^{-2t} & te^{-2t} \\ e^t & -2e^{-2t} & e^{-2t}-2te^{-2t} \\ e^t & 4e^{-2t} & -4e^{-2t}+4te^{-2t} \end{vmatrix}$$

$$= e^t\left[-2e^{-2t}(-4e^{-2t}+4te^{-2t}) - (4e^{-2t})(e^{-2t}-2te^{-2t})\right] - e^{-2t}\left[e^t(-4e^{-2t}+4te^{-2t}) - e^t(e^{-2t}-2te^{-2t})\right] + te^{-2t}\left[e^t \cdot 4e^{-2t} + 2e^te^{-2t}\right]$$

missing

$$= e^t\left[8e^{-4t} + 4te^{-4t}\right] - e^{-2t}\left[-4e^{-t}+4te^{-t}-e^{-t}+2te^{-t}\right] + te^{-2t}\left[4e^{-t}+2e^{-t}\right] = 8e^{-3t} \cancel{+4t} = \cdots \neq 0.$$

$y = k_1e^t + k_2e^{-2t} + k_3te^{-2t} \; ; \; y(0)=0 \Rightarrow k_1+k_2=0.$

$y' = k_1e^t - 2k_2e^{-2t} + k_3e^{-2t} - 2k_3te^{-2t} \; ; \; y'(0)=2 \Rightarrow k_1-2k_2+k_3=2$

$y'' = k_1e^t + 4k_2e^{-2t} - 2k_3e^{-2t} - 2k_3e^{-2t} + 4k_3te^{-2t} \; ; \; y''(0)=8 \Rightarrow k_1+4k_2-4k_3=8$

$k_1+k_2=0 \Rightarrow k_2=-k_1$

$k_1-2k_2+k_3=2 \Rightarrow 3k_1+k_3=2$

$k_1+4k_2-4k_3=8 \Rightarrow -3k_1-4k_3=8$

$+ \therefore -3k_3=8 \Rightarrow \boxed{k_3=-\frac{8}{3}}$; $k_1=\frac{2-k_3}{3}=\boxed{\frac{14}{9}}$; $\boxed{k_2=-\frac{14}{9}}$.

6. (12%) Consider the following initial-value problem:

$$t^2\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-12y=0; y(1)=0; y'(1)7; t>0.$$

Let $x=\ln t$; use this change of variables to find the particular solution to this IVP.

$$x=\ln t \Rightarrow \frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=\frac{dy}{dx}\cdot\frac{1}{t}$$

$$\frac{d^2y}{dt^2}=\left(\frac{d^2y}{dx^2}\cdot\frac{1}{t}\right)\cdot\frac{1}{t}-\frac{1}{t^2}\cdot\frac{dy}{dx}$$

$$=\frac{1}{t^2}\frac{d^2y}{dx^2}-\frac{1}{t^2}\frac{dy}{dx}.$$

Substitute: $t^2\left(\frac{1}{t^2}\frac{d^2y}{dx^2}-\frac{1}{t^2}\frac{dy}{dx}\right)+2t\left(\frac{1}{t}\frac{dy}{dx}\right)-12y=0.$

$$\therefore \frac{d^2y}{dx^2}-\frac{dy}{dx}+2\frac{dy}{dx}-12y=0$$

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=0 \Rightarrow r^2+r-12=0$$

$$(r+4)(r-3)=0 \left\langle \begin{array}{l} r=-4 \\ r=3 \end{array}\right.$$

$$\therefore y_1=e^{-4x}=e^{-4\ln t}=1/t^2$$

$$y_2=e^{3x}=e^{3\ln t}=t^3.$$

$$\therefore y=\frac{k_1}{t^2}+k_2t^3;\quad y(1)=0 \Rightarrow k_1+k_2=0 \quad \times 2$$

$$y'=\frac{-2k_1}{t^3}+3k_2t^2;\quad y'(1)=7 \Rightarrow -2k_1+3k_2=7 \quad +$$

$$5k_2=7 \Rightarrow \boxed{k_2=\frac{7}{5}}$$

and $\boxed{k_1=-7/5}$.