LAU
School of **Arts and Sciences**

Department of Computer Science & Mathematics

MTH207 – Discrete Mathematics
Spring 2016
Exam 3
(April 23, 2016)

Name: Answer Key ID:

Duration: 60 minutes

- Answer the questions in the space provided for each problem.
- If more space or scratch is needed, you may use the back pages.
- Only scientific calculators are permissible.
- The exam has **7** pages consisting of **11** exercises.

Grades:

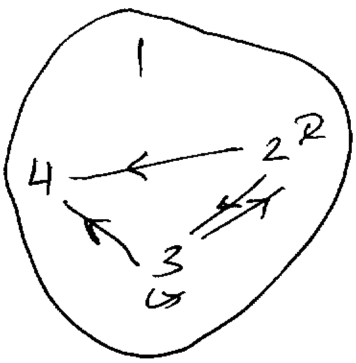
N1	**N2**	**N3**	**N4**	**N5**	**N6**	**N7**	**N8**
(15%)	(6%)	(10%)	(12%)	(12%)	(10%)	(15%)	(6%)

N9	**N10**	**N11**	**Total**
(4%)	(4%)	(6%)	(100%)

($\frac{1}{2}$) property ($\frac{3}{4}$) justification

1. (15 %) For each of these relations on the set {1, 2, 3, 4}, decide whether it is **reflexive**, whether it is **symmetric**, whether it is **antisymmetric**, and whether it is **transitive**. Justify your answer.

   a) $R_1 = \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$



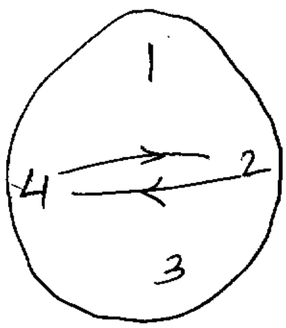
Not reflexive $1\frac{1}{4}$ $(1,1) \notin R$

Not Symmetric $1\frac{1}{4}$ $(2,4) \in R$ but $(4,2) \notin R$

Not Antisymmetric $1\frac{1}{4}$ $(2,3)$ and $(3,2) \in R$ but $2 \neq 3$

Transitive $1\frac{1}{4}$

   b) $R_2 = \{(2, 4), (4, 2)\}$



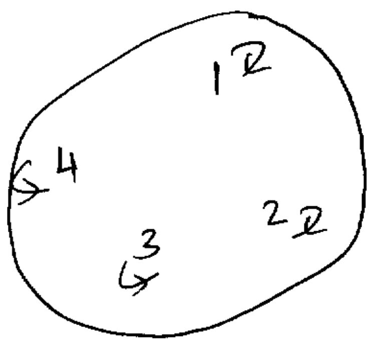
Not Reflexive $1\frac{1}{4}$ $(1,1) \notin R$

Symmetric $1\frac{1}{4}$ $(2,4)$ & $(4,2) \in R$

Not Antisymmetric $1\frac{1}{4}$ $(2,4)$ & $(4,2) \in R$ but $2 \neq 4$

Not transitive $1\frac{1}{4}$ $(2,4)$ & $(4,2) \in R$ but $(2,2) \notin R$

   c) $R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$



Reflexive $1\frac{1}{4}$

Symmetric $1\frac{1}{4}$

Antisymmetric $1\frac{1}{4}$

Transitive $1\frac{1}{4}$

2. (6 %) Consider the relations R and S on {1, 2, 3, 4}
R = {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}
S = {(1,2), (1,3),(2,1),(4,3)}
Find RoS.

RoS = { (1,2), (1,3), (2,1), (2,4), (4,3) }

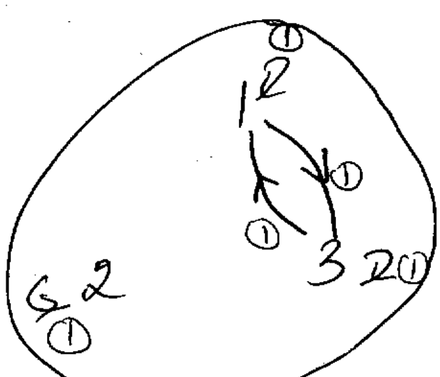
SoR = { (1,2), (1,3), (2,1), (4,2), (4,3) }

3. a) (5 %) **List the ordered pairs** in the relation R on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

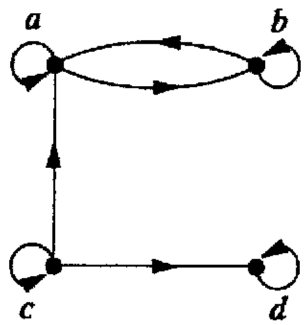
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

R = { (1,1), (1,3), (2,2), (3,1), (3,3) }

b) (5%) Draw the **directed graph** representing the relation in part (a)



4. (12%) Determine whether the relation represented by the directed graph below is **reflexive, symmetric, antisymmetric,** and/or **transitive**. Justify your answer.



Reflexive 1 – all loops 2

Not Symmetric 1 – $(c,a) \in R$ but $(a,c) \notin R$ 2

Not Antisymmetric 1 – (a,b) and $(b,a) \in R$ but $a \neq b$ 2

Not transitive 1 – $(c,a) \wedge (a,b) \in R$ but $(c,b) \notin R$ 2

5. (12 %) Let **A** be the set of all integers. Define the relation *R* on A by:

R = {(a, b) ∈**A** | a ≤ b}

Is (A,R) a **POSET**?

- $\forall a \quad a \leq a$ true $\Rightarrow (a,a) \in R$ 2
  Reflexive 2
- $\forall a,b$ if $\underbrace{(a,b) \in R}_{a \leq b}$ and $\underbrace{(b,a) \in R}_{b \leq a}$ 2
  then $a = b$
  Antisymmetric 2
- $\forall a,b,c$
  if $\underbrace{(a,b) \in R}_{a \leq b}$ and $\underbrace{(b,c) \in R}_{b \leq c}$ 2
  then $a \leq c$
  $\Rightarrow (a,c) \in R$
  Transitive 2

Reflexive, antisymmetric and transitive

$\Rightarrow$ POSET.

6. (5+5= 10 %) Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{M}_{R2} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

a) Find the **matrix** that represent $R_1 \cup R_2$

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2 3

b) **Is** the relation $R_1 \cup R_2$ an **equivalence relation**? Justify.

diagonal $\neq 1$ 2

$\Rightarrow$ not reflexive $1\frac{1}{2}$

$\Rightarrow$ not equivalence relation $1\frac{1}{2}$

7. (9+6=15 %) Let $A=Z$ and let R denote the relation on A defined by :

aRb if $a^2 \equiv b^2 \ (mod\ 3)$ $\longrightarrow$ $\boxed{a^2 \bmod 3 = b^2 \bmod 3}$

a. **Show** that $\boldsymbol{R}$ is an **equivalence relation.**

- $\forall a$ $\quad a^2 \bmod 3 = a^2 \bmod 3$

  $\Rightarrow (a,a) \in R$ 2

  $\Rightarrow$ Reflexive 1

- $\forall a,b$ $\quad (a,b) \in R \Rightarrow a^2 \bmod 3 = b^2 \bmod 3$

  $\Rightarrow b^2 \bmod 3 = a^2 \bmod 3$ 2

  $\Rightarrow (b,a) \in R$

  $\Rightarrow$ symmetric 1

- $\forall a,b,c$ $\quad (a,b) \in R \wedge (b,c) \in R$

  $\Rightarrow a^2 \bmod 3 = b^2 \bmod 3 \wedge b^2 \bmod 3 = c^2 \bmod 3$

  $\Rightarrow a^2 \bmod 3 = c^2 \bmod 3$ 2

  $\Rightarrow (a,c) \in R$

  $\Rightarrow$ Transitive 1

R is reflexive, symmetric, and transitive hence R is equivalence relation

b. Find <u>all</u> **equivalence classes** of *R*.

1 $a^2 \bmod 3 = 0, 1, 2$

1 $0^2 \equiv 0 \pmod 3$

1 $1^2 \equiv 1 \pmod 3$

1 $2^2 \equiv 1 \pmod 3$ $= \frac{1}{2}$

2 distinct classes

$1\frac{1}{2}$ $[0]$ and $[1] = [2]$

or

- If $a = 3k$ then

$1\frac{1}{2}$ $a^2 = (3k)^2 = 9k^2 = 3(3k^2)$

$\boxed{a^2 \bmod 3 = 0}$

- If $a = 3k+1$ then

$a^2 = (3k+1)^2 = 9k^2 + 6k + 1$

$1\frac{1}{2}$ $= 3(3k^2 + 2k) + 1$

$\boxed{a^2 \bmod 3 = 1}$

- If $a = 3k+2$ then

$1\frac{1}{2}$ $a^2 = (3k+2)^2 = 9k^2 + 12k + 4$

$= 9k^2 + 12k + 3 + 1$

$= 3(3k^2 + 4k + 1) + 1$

$\boxed{a^2 \bmod 3 = 1}$

hence, 2 distinct classes $[0]$ and $[1]$ $1\frac{1}{2}$

8. (6 %) How many strings of four decimal digits
   a. Do not contain the same digit twice?

$\underline{10} \times \underline{9} \times \underline{8} \times \underline{7} = 5040$

3

b. End with an even digit?

$\underline{10} \times \underline{10} \times \underline{10} \times \underline{5} = 5000$

3

5 even digits $\Rightarrow 0, 2, 4, 6, 8$

9. (4%) If there are 30 students in a class, then at least ______ have last names that begin with the same letter.

pigeonhole $\left\lceil \frac{30}{26} \right\rceil = 2$

2 2

10. (2+2=4%) How many bit strings of length 12 contain

a. Exactly three 1s?

Three 1s
Nine 0s $\Rightarrow$ permute 111000000000

$$\frac{12!}{3!\,9!} = 220 \quad 2$$

b. An equal number of 0s and 1s

Six 0s
Six 1s $\Rightarrow$ permute 000000111111

$$\frac{12!}{6!\,6!} = 924 \quad 2$$

11. (3+3=6%) A club has 25 members.

a. How many ways are there to choose four members of the club to serve on an executive committee?

$$C(25,4)^{1} = \frac{25!}{4!\,21!}^{1} = 12650^{1}$$

$\binom{25}{4}$; ${}_{25}C_4$; $C_{25,4}$; C^{25}_{4} ; $C(25,4)$ all accepted

b. How many ways are there to choose a president, vice president, secretary, and treasurer of the club?

$$\underline{25} \times \underline{24} \times \underline{23} \times \underline{22} = \overset{1}{303600}$$

$$\underset{1}{P(25,4)} = \frac{25!}{\underset{1}{(25-4)!}} = \frac{25!}{21!}$$