LAU
School of Arts and Sciences

Department of Computer Science & Mathematics

MTH207 – Discrete Mathematics
Spring 2016
Exam 2
(March 21, 2016)

Name: Answer Key ID:

Duration: 50 minutes

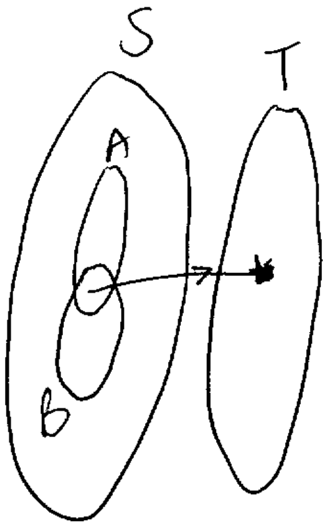
- Answer the questions in the space provided for each problem.
- If more space or scratch is needed, you may use the back pages.
- Only scientific calculators are permissible.
- The exam has **6** pages consisting of **7** exercises.

Grades:

1. 6%	
2. 10%	
3.a 8%	
3.b 8%	
3.c 8%	

4. 8%	
5. 12%	
6.a, b 16%	
7.a,b,c 24%	
Total 100%	

1. (6%)) Let f be a one-to-one function from the set S to the set T. Let A and B be subsets of S. Prove that $f(A) \cap f(B) \subseteq f(A \cap B)$



suppose $x \in f(A) \cap f(B)$ 1

$\Rightarrow x \in f(A)$ and $x \in f(B)$ 1

$\exists\, a \in A$ such that $f(a) = x$ ½ $\qquad \exists\, b \in B$ ½ such that $f(b) = x$

but since f is 1-1 then $a = b$ 1

$a \in A$ and $a \in B$ since $a = b$ $\Rightarrow$ $a \in A \cap B$ 1

$\underbrace{f(a)}_{x} \in f(A \cap B)$

$x \in f(A \cap B)$ ½

$\Rightarrow f(A) \cap f(B) \subseteq f(A \cap B)$ ½

2. (10%) If A, B and C are countable sets, show that (AxB)UC is also countable.

1 If $A \times B$ is countable then $(A \times B) \cup C$ is countable

(Theorem - union of 2 countable sets is countable)

1 hence we need to show that $A \times B$ is countable

Both A and B are countable $\Rightarrow$ 3 cases

1) 1 Both A and B are finite sets

$A = \{a_1, a_2, \dots a_m\}$ $\quad B = \{b_1, b_2, \dots b_n\}$

$|A| = m$ $\quad |B| = n$

1 $|A \times B| = |A| \cdot |B| = m \cdot n \Rightarrow A \times B$ is finite

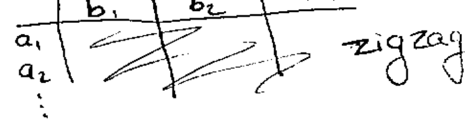
2) 1 One is finite (suppose A) and the other is infinite countable (suppose B)

$A = \{a_1, a_2, \dots a_m\}$ ½ $\quad B = \{b_1, b_2, \dots\}$ ½

1 $A \times B = \{(a_1, b_1), (a_2, b_1) \dots (a_m, b_1), (a_1, b_2), (a_2, b_2) \dots (a_m, b_2), \dots\}$ infinite countable

3) 1 both A and B are infinite countable

$\Rightarrow A \times B$ countable 2



3. a)(8%) If $b_n = 3 + b_{n-1}$, for all $n \geq 1$, if $b_0 = 7$, show that $b_n = 7 + 3n$ for all nonnegative integers n.

$$b_1 = 3 + b_0$$

$$b_2 = 3 + b_1 = 3 + (3 + b_0) = 3 \times 2 + b_0$$

$$b_3 = 3 + b_2 = 3 + (3 \times 2) + b_0 = 3 \times 3 + b_0$$

$$\vdots$$

$$b_n = 3 \times n + \boxed{b_0} \rightarrow \boxed{b_n = 3n + 7}$$

$b_0 \to 7$

b)(8%) Give as good a big-O estimate as possible for the function
$f(x) = (2\log n! + \sqrt{n+1})(n^{3/2} + 2n)$

$$\underbrace{O(n \log n) \quad O(n^{1/2})}_{O(n \log n)} \qquad \underbrace{O(n^{3/2}) \quad O(n)}_{O(n^{3/2})}$$

$$O(n^{5/2} \log n)$$

c)(8%) Give a big-O estimate for the function $\sum_{k=1}^{n} k(k+1)$

$$\sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n} (k^2 + k)$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$\underbrace{O(n^3) \qquad O(n^2)}$$

$$= O(n^3)$$

4. (8%) Find the Boolean product of A and A^T, where $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad 3$$

$$A \odot A^T = \underset{3\times 4}{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \odot \underset{4\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \quad 2 \text{ order}$$

$$= \underset{3\times 3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} \quad 3$$

5. a) (12%) Let $f: N \times N \to Z \times Z$ be defined by

$$f(m,n) = (3m + n, n^2)$$

i. Is f 1 – 1 ? Justify your answer.

$\left.\begin{matrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{matrix}\right\} \in \mathbb{N}$

1 suppose $f(m_1, n_1) = f(m_2, n_2)$

1 $(3m_1 + n_1, n_1^2) = (3m_2 + n_2, n_2^2)$

1 $n_1^2 = n_2^2$

1 $\boxed{n_1 = n_2}$

and 1 $3m_1 + \cancel{n_1} = 3m_2 + \cancel{n_2}$

$3m_1 = 3m_2$

1 $\boxed{m_1 = m_2}$

hence f is 1-1

ii. Is f onto? Justify your answer.

2 f is not onto

2 since $(0,-1) \in \mathbb{Z} \times \mathbb{Z}$ has no pre-image

$(m,n) \in \mathbb{N} \times \mathbb{N}$ such that

$f(m,n) = (3m+n, n^2)$

2 $n^2 \neq -1$

6. a. (8%) Use the **Euclidean Algorithm** to find the $gcd(245, 175)$.

$$245 = 175 \times \underline{1} + \underline{70} \quad 2$$
$$175 = 70 \times \underline{2} + \underline{35} \quad 2$$
$$70 = 35 \times \underline{2} + \underline{0} \quad 2$$
$$GCD = 35 \quad 2$$

b. (8 %) Express the GCD(245,175) as a linear combination of 245 and 175.

$a = 245 \qquad b = 175 \qquad c = 35$

$$\begin{cases} a = b + 70 \; ① \\ b = 70 \times 2 + GCD \; ① \end{cases} \begin{array}{l} \Rightarrow 70 = a - b \; ① \\ \Rightarrow GCD = b - (70 \times 2) \; ① \end{array}$$
$$= b - (a-b) \times 2 \; ①$$
$$= b - 2a + 2b \; ①$$
$$= 3b - 2a \; ①$$
$$35 = 3(175) - 2(245) \; ①$$

7. (8%) a. Suppose $a \equiv 5 \ (mod\ 24)$ and $b \equiv 13 \ (mod\ 24)$. Find the integer c between 0 and 24 such that $c \equiv 2a^2 + 3b \ (mod\ 24)$

Method 1

$$a \equiv 5 \pmod{24}$$
$$a^2 \equiv 25 \pmod{24}$$
$$2a^2 \equiv 50 \pmod{24}$$
$$+\ 3b \equiv 39 \pmod{24}$$
$$c \equiv 89 \pmod{24}$$
$$\boxed{c = 17}$$

Method 2

$$a \equiv 5 \pmod{24}$$
$$b \equiv 13 \pmod{24}$$
$$0 < c < 24$$
$$c \equiv 2a^2 + 3b \pmod{24}$$

$$\left.\begin{array}{l} ② a = 5 \\ ② b = 13 \end{array}\right] \Rightarrow ② \ c \equiv 89 \pmod{24}$$

$\Rightarrow 89 - c$ is multiple of 24

$2a^2 + 3b$
$= 50 + 39$
$= 89$

$\Rightarrow ② \ \boxed{c = 17}$

Method 3

$$a = 5 + 24m_1$$
$$b = 13 + 24m_2$$

$$a^2 = (5 + 24m_1)^2 = 25 + 10(24m_1) + (24m_1)^2$$
$$c = 2a^2 + 3b = \underline{50} + 20(24m_1) + 2(24m_1)^2 + \underline{3(13)} + 3(24m_2)$$
$$= 89 + 24[20m_1 + 2(24m_1^2) + 3m_2]$$
$$c \equiv 89 \pmod{24} \quad \Rightarrow \boxed{c = 17}$$

(8%) b. Show that if a, b, k and m are integers such that $k \geq 1, m \geq 2$, and $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$.

$$\begin{array}{l} a \equiv b \pmod{m} \\ \times\ a \equiv b \pmod{m} \\ \hline a^2 \equiv b^2 \pmod{m} \end{array}$$

$$\begin{array}{l} a^2 \equiv b^2 \pmod{m} \\ \times\ a \equiv b \pmod{m} \\ \hline a^3 \equiv b^3 \pmod{m} \end{array}$$

$\vdots$

$$\boxed{a^k \equiv b^k \pmod{m}}$$

$\boxed{\text{OR}}$

$a \equiv b \pmod{m}$

$a - b = cm$ 2

$a = b + cm$ 1

$a^k = (b + cm)^k$ 1

$= b^k + \underbrace{k b^{k-1} cn + \cdots}_{\text{multiples of } n}$ 1

$= b^k + nQ$ 1

$a^k - b^k = nQ$ 1

$a^k \equiv b^k \pmod{m}$ 1

(8%) c. Prove that the product of any three consecutive integers is divisible by 6.

3 consecutive integers : $n, \; n+1, \; n+2$

* Product is divisible by 2

→ If n is odd then $n = 2k+1$

$1\frac{1}{2}$ $n(n+1)(n+2) = (2k+1)(2k+2)(2k+3)$

$= (2k+1)\, 2(k+1)(2k+3)$ even

→ If n is even then $n = 2k$

$1\frac{1}{2}$ $n(n+1)(n+2) = 2k(2k+1)(2k+2)$ even

* Product is divisible by 3

→ If n is divisible by 3 then $n = 3k$

1 $n(n+1)(n+2) = 3k(3k+1)(3k+2)$ divisible by 3

→ 1 If n is not divisible by 3 then

$n = 3k+1$ or $n = 3k+2$

$n = 3k+1$ 1

$n(n+1)(n+2)$

$= (3k+1)(3k+2)(3k+3)$

$= (3k+1)(3k+2)\,3(k+1)$

divisible by 3

$n = 3k+2$

$n(n+1)(n+2)$ 1

$= (3k+2)(3k+3)(3k+4)$

$= (3k+2)\,3(k+1)(3k+4)$

divisible by 3

hence

1 $n(n+1)(n+2)$ is divisible by both 2 and 3

hence divisible by 6.