

KEY

MTH 207. Exam II.
Fall 2012

Instructions

- Write your name and ID.
 - NAME:
 - ID:

- ANY ATTEMPT in cheating or using external sources such as notes, cell phones and graphic calculators will result in being expelled from the room, getting an automatic zero and further disciplinary actions.
- You have 60 minutes to finish this exam.
- There are 5 problems in this exam (check that you have 8 pages).
- Show all your work (to get partial credit).

PROBLEM I [20 Pts]

1. Let A and B be two sets. Let $f : A \times B \rightarrow A$ be given by

$$f(a, b) = a, \text{ for any } a \in A, b \in B.$$

(this function picks the first element from each pair, it is called a projection)
Is f one to one? (Hint: what if the set A is made of just one element?)

This function is not 1-1 in general. We prove this fact by a counterexample.

Let $A = \{a_1\}$. Then even though $f(a_1, b_1) = a_1 = f(a_1, b_2)$, $(a_1, b_1) \neq (a_1, b_2)$ for $b_1 \neq b_2 \in B$.

Now If it happens that $B = \{b_1\}$, then the function becomes 1-1. Proof: Let $(a_1, b_1), (a_2, b_2)$ be in $A \times B$, then must have $a_1 = a_2 \Rightarrow (a_1, b_1) = (a_2, b_2)$ $\boxed{\begin{array}{l} b_1 = b_2 \\ (B \text{ one elt}) \end{array}}$

2. Find the inverse of each of these functions (you don't have to show that the function is bijective first):

- $f : (0, \infty) \rightarrow (0, \infty), f(x) = 1/x$
- $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, f(x, y) = (x + y, x - y)$

• let $y = f(x) = 1/x$ solve for x
 $x = 1/y, y \neq 0 \Rightarrow f^{-1}(x) = \frac{1}{x}, x \neq 0$ its own inverse
 $f^{-1} : (0, \infty) \rightarrow (0, \infty)$

• let $(z_1, z_2) = f(x, y) = (x + y, x - y)$
 We need to express x as a function of z_1 & z_2
 and y as a function of z_1 & z_2

$$\Rightarrow \begin{cases} z_1 = x + y \\ z_2 = x - y \end{cases} \text{ solve for } x \text{ & } y$$

$$\begin{aligned} z_1 + z_2 &= 2x & \Rightarrow x &= \frac{z_1 + z_2}{2} \\ z_1 - z_2 &= 2y & \Rightarrow y &= \frac{z_1 - z_2}{2} \end{aligned} \Rightarrow f^{-1}(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2} \right)$$

Exercise: check $f \circ f^{-1}(x, y) = (x, y)$ $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$

PROBLEM II [20 Pts]

1. Compute the sum $1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{13}}$

$$a_n = \left(\frac{1}{2}\right)^n, n=0, 1, 2, \dots$$

This is a geometric progression. We are taking the finite sum of the first 14 terms.

$$S_{14} = \frac{1 - \left(\frac{1}{2}\right)^{14}}{1 - \frac{1}{2}} = 2\left(1 - \left(\frac{1}{2}\right)^{14}\right)$$

2. Compute the product $\prod_{j=1}^m \frac{j+1}{j}$

$$\begin{aligned} \prod_{j=1}^m \frac{j+1}{j} &= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \dots \cdot \frac{m}{m-1} \cdot \frac{m+1}{m} \\ &= m+1 \end{aligned}$$

3. Find a formula for the n^{th} term of the sequence

$$1, 4, 7, 10, \dots$$

$$S_n = 3n + 1 \quad n = 0, 1, 2, \dots$$

PROBLEM III [20 Pts]

- Find the smallest possible number r such that $f(n) = \sqrt{n^2+1}(2n^4 - 4)$ is $O(n^r)$. Justify your answer.

Thoughts $\sqrt{n^2+1} \sim n$
 $2n^4 - 4 \sim n^4$ so overall it should be
 $O(n^5)$

Now we prove it:

$$\sqrt{n^2+1} \leq \sqrt{2n^2}$$
 since $n^2+1 \leq 2n^2$
 $= \sqrt{2} \cdot n$ since $\sqrt{-n^2} \leq -1$
since $n^2 \geq 1$
 $\underline{n \geq 1}$

also $2n^4 - 4 \leq 2n^4 \forall n$

$$\Rightarrow \text{multiply, get } f(n) \leq 2\sqrt{2}n^5 \text{ for } n \geq 1$$

$$\Rightarrow f(n) \text{ is } O(n^5)$$

- Prove that $\sum_{k=1}^n k^2$ is $O(n^3)$ (You don't have to remember any formulas)

$$\begin{aligned}
 S_n &= \sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 \\
 &\leq \underbrace{n^2 + n^2 + \dots + n^2}_{\text{adding same } n \text{ times}} = n(n^2) = n^3
 \end{aligned}$$

$\Rightarrow C = 1 \quad K = 0$

- (+5) Bonus: find a ratio of polynomials $g(n)$ such that $f(n) = \frac{\ln(n!)}{n^4}$ is $O(g(n))$. Justify.

$$\begin{aligned}
 \ln(n!) &= \ln(n(n-1)(n-2)\dots 2 \cdot 1) \\
 &= \ln n + \ln(n-1) + \ln(n-2) + \dots + \ln(2) + \ln(1) \\
 &\leq n + n + \dots + n = n^2
 \end{aligned}$$

$\Rightarrow \frac{\ln(n!)}{n^4} \leq \frac{n^2}{n^4} = \frac{1}{n^2} \quad \forall n > 0$

$\textcircled{K=0, C=1}$

PROBLEM IV [16 Pts]

1. If $n \bmod p = 0$, show that $-n \bmod p = 0$

$$\begin{aligned} n \bmod p = 0 \text{ means } n &= pk \text{ for some } k \in \mathbb{Z} \\ \Rightarrow -n &= -pk = K'p \quad \text{where} \\ \Rightarrow -n \bmod p &= 0 \quad K' = -k \in \mathbb{Z} \end{aligned}$$

2. If $m \equiv n \pmod{p}$, show that $m^2 \equiv n^2 \pmod{p}$

$$\text{we have } m - n = kp \quad k \in \mathbb{Z}$$

multiply both sides by $m+n \in \mathbb{Z}$

$$\begin{aligned} (m+n)(m-n) &= (m+n)kp \\ m^2 - n^2 &= K''p \quad K'' \in \mathbb{Z} \end{aligned}$$

$\Rightarrow m^2 - n^2$ are equiv mod p

$$\Rightarrow m^2 \equiv n^2 \pmod{p}$$

3. Find $\gcd(45, 33)$

$$45 = 3 \times 15 = 3 \times 3 \times 5$$

$$33 = 3 \times 11$$

$$\Rightarrow \gcd(45, 33) = 3$$

PROBLEM V [24 Pts]

1. Let \mathbf{A}, \mathbf{B} be two matrices of size $m \times n$. If you know that $\mathbf{A}^T = \mathbf{B}^T$, does that mean that $\mathbf{A} = \mathbf{B}$? Justify.

If $\mathbf{A}^T = \mathbf{B}^T \Rightarrow \mathbf{A}^T - \mathbf{B}^T = \mathbf{0}$ <sup>the zero
 $m \times n$
matrix</sup>

$$\Rightarrow (\mathbf{A} - \mathbf{B})^T = \mathbf{0}$$

because you can easily prove that $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

$$\Rightarrow ((\mathbf{A} - \mathbf{B})^T)^T = \mathbf{0}^T = \mathbf{0} \Rightarrow \mathbf{A} - \mathbf{B} = \mathbf{0}$$
$$\Rightarrow \boxed{\mathbf{A} = \mathbf{B}}$$

You can also write the ~~non-zero~~ entries of each \mathbf{A} & \mathbf{B} and show result.

2. Let \mathbf{C} be the 3×3 matrix given by $\mathbf{C} = \begin{pmatrix} 1 & 3 \\ 2 & -4 \\ 5 & -2 \end{pmatrix}$. Compute \mathbf{C}^T and \mathbf{C}^2 .

$$\mathbf{C}^2 = \begin{pmatrix} 2 & -7 & 2 \\ -1 & 20 & -6 \\ 1 & 23 & -7 \end{pmatrix}$$

$$\mathbf{C}^T = \begin{pmatrix} 1 & 2 & 5 \\ 3 & -4 & -2 \\ -1 & 1 & 0 \end{pmatrix}$$

3. Let \mathbf{A} be the 2×2 matrix given by $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, where a is any real number. Let \mathbf{B} be any 2×2 matrix. Show that $\mathbf{AB} = \mathbf{BA}$.

$$\text{Let } \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad \begin{array}{l} b_{ij} \in \mathbb{R} \\ i=1, 2 \\ j=1, 2 \end{array}$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} ab_{11} & ab_{12} \\ ab_{21} & ab_{22} \end{pmatrix}$$

$$\mathbf{B} \cdot \mathbf{A} = \begin{pmatrix} b_{11}a & b_{12}a \\ b_{21}a & b_{22}a \end{pmatrix} = AB \text{ because ordinary multiplication is commutative.}$$