KEY

MTH 207. Exam I

Fall 2012

Instructions

- Write your name and ID.
  - NAME:
  - ID:

- ANY ATTEMPT in cheating or using external sources such as notes, cell phones and graphic calculators will result in being expelled from the room, getting an automatic zero and further disciplinary actions.
- You have 75 minutes to finish this exam.
- There are 4 problems in this exam (check that you have 12 pages). The exam is out of 100 but there are 5 BONUS points, so the maximum achievable score is 105. Leave the bonus problems to the END.
- Show all your work (to get partial credit).

Problem I

1. (6%) If you know that the proposition $p \to q$ is false, what is the truth value of

   (a) $p \vee q$

   (b) $q \to p$

   (c) $p \wedge q$

If $p \to q$ is false

then p is T and q is F

(a) $p \vee q$ T or F gives $\boxed{T}$

(b) $q \to p$ is a true impticatun $\boxed{T}$ Since False can imply anything

(c) $p \wedge q$ T $\wedge$ F gives $\boxed{F}$

2. (12%) Is $\forall x(P(x) \to Q(x))$ equivalent to $(\forall x P(x)) \to (\forall x Q(x))$? Justify your answer.

The answer is No.

There are many ways to give counterexamples

Counter example 1:

Let $P(x)$ be any predicate which is sometimes true sometimes false.

Let $Q(x)$ be a predicate which is always false

($\forall x\, Q(x)$ false)

Then $\forall x\,(P(x) \to Q(x))$ is false but

$\forall x\, P(x) \to \forall x\, Q(x)$ is true

you need to think why we can say that?

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Counter example 2:

Let the domain of x consist of $\{x_1, x_2\}$

Then $\forall x\,(P(x) \to Q(x)) \Leftrightarrow$

$\boxed{P(x_1) \to Q(x_1) \wedge P(x_2) \to Q(x_2)}$ (1)

But $\forall x P(x) \to \forall x Q(x) \Leftrightarrow$

$\boxed{P(x_1) \wedge P(x_2) \to Q(x_1) \wedge Q(x_2)}$ (2)

(1) is not equivalent to (2) [truth Tables]

Problem II

Prove the following. Name the type of proof you are using.

(a) (12%) Show that $n^2 - 2$ is never divisible by 3 for $n \in \mathbb{N}$
(Hint: go through all the possibilities, $n = 3k$, $3k + 1$ or $3k + 2$, for $k \in \mathbb{N}$)

This is an exhaustive proof.

Case 1 $n = 3k$

Then $n^2 - 2 = \underbrace{9k^2}_{\text{multiple of } 3} - 2 \Rightarrow$ obviously not divisible by 3

Case 2 $n = 3k+1$

$n^2 - 2 = (3k+1)^2 - 2 = 9k^2 + 6k + 1 - 2$

$= \underbrace{9k^2 + 6k}_{\text{multiple of } 3} - 1$

$\Rightarrow$ not divisible by 3

Case 3: $n = 3k+2$

$n^2 - 2 = (3k+2)^2 - 2$

$= 9k^2 + 4 + 6k - 2$

$= \underbrace{9k^2 + 6k}_{\text{multip of } 3} + 2 \Rightarrow$ not divisible by 3

$\Rightarrow$ In all cases, not divisible by 3 $\square$

(b) (12%) Show that $\sqrt{3}$ is irrational

We shall prove it by contradiction.

Suppose $\sqrt{3}$ is rational

then $\sqrt{3} = \frac{m}{n}$ where m & n are integers with no common divisors $n \neq 0$

then $3 = m^2/n^2$

then $m^2 = 3n^2$ which means m^2 is a multiple of 3 $\Rightarrow$ (To Prove!) m is a multiple of 3.

$\Rightarrow m = 3k \Rightarrow n^2 = m^2/3 = 9k^2/3 = 3k^2$

$\Rightarrow n^2$ is a multiple of 3 $\Rightarrow$ n is a multiple of $\boxed{3}$ $\Rightarrow$ m and n have a common factor $\rightarrow\leftarrow$

To Prove: If m^2 is a multiple of 3 then m is a multiple of 3

$(p \rightarrow q)$

Let's prove $\bar{q} \rightarrow \bar{p}$ (contrapositive)

Suppose m is Not a multiple of 3,

then $m = 3k+1$ or $m = 3k+2$

then $m^2 = \underbrace{9k^2+6k}_{\text{m of 3}}+1$ or $m^2 = 9k^2+4+12k$ $= (9k^2+3+12k)+1$

$\Rightarrow m^2$ not a multiple of 3 not a multiple of 3

$n \in \mathbb{P}$

(c) (12%) Show that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

$P(1) \equiv 1 \geq 1$ ✓

$P(k) = 1 + \frac{1}{\sqrt{2}} + - + \frac{1}{\sqrt{k}} \geq \sqrt{k}$

Suppose $P(k)$ is true

Then $1 + \frac{1}{\sqrt{2}} + - + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}}$

meaning $1 + \frac{1}{\sqrt{2}} + -- + \frac{1}{\sqrt{k+1}} \geq \frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}}$

(since $k+1 > k$) $\geq \frac{\sqrt{k \cdot k} + 1}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}}$ rationed

$= \frac{\cancel{(k+1)}\sqrt{k+1}}{\cancel{k+1}}$

We just proved $P(k+1)$

$n \geq 2$

2

(d) (+5%) **Bonus: show that $5^n - 4n - 1$ is divisible by 16.**

$5^n - 4n - 1$ $n \geq 2$

$P(2) = 5^2 - 4.2 - 1$
$= 16$ obvious

$25 - 4.2 - 1$
$25 - 8 - 1$
$25 -$

Suppose $P(k)$ holds,
meaning $5^k - 4k - 1$ is divisible by 16
then n.t.s $5^{k+1} - 4(k+1) - 1$ //

$5^{k+1} - 4(k+1) - 1 = 5^k.5 - 4k - 5$
$= 5^k.5 - 20k + 16k - 5$
$= 5\underbrace{(5^k - 4k - 1)}_{\text{multiple of 16 by asumption } P(k)} + \underbrace{16k}_{\text{multiple of 16.}}$

$\Box$

Problem III

Prove the following set relations (you cannot use Venn Diagrams)

(a) (8%) $\overline{(A \cap B \cap C)} = \overline{A} \cup \overline{B} \cup \overline{C}$

let $D = A \wedge B$

Apply Demorgan's for 2 sets

$$\overline{D \wedge C} = \overline{D} \cup \overline{C}$$

Rewrite D as $A \wedge B$ & apply De morgan's again

$$\overline{A \wedge B \wedge C} = \overline{A \wedge B} \cup \overline{C}$$

$$= \overline{A} \cup \overline{B} \cup \overline{C}$$

QED

(b) (8%) $(A \subset B) \leftrightarrow (\overline{B} \subset \overline{A})$

① $\longrightarrow$

Assume $A \subset B$, Prove $\overline{B} \subset \overline{A}$

$\underbrace{\text{for every } x \in A}_{p} \Rightarrow \underbrace{x \in B}_{q}$ (propositions)

Apply contrapositive for propositions

$\overline{q} \Rightarrow \overline{p}$

$\Rightarrow$ for every $x \in \overline{B} \Rightarrow x \notin A$

$x \in \overline{A}$

$\Rightarrow \overline{B} \subset \overline{A}$

② $\longleftarrow$ Need to show if $x \in \overline{B} \Rightarrow x \in \overline{A}$

and $x \in A$ then $x \in B$

very similar

Problem IV

Let $f : S \to T$ denote a function.

(a) (6%)Which of the following statements mean that the function is one to one?

For any $s_1, s_2 \in S$

(i) If $f(s_1) = f(s_2)$ then $s_1 = s_2$

(ii) If $s_1 = s_2$, then $f(s_1) = f(s_2)$

(iii) If $s_1 \neq s_2$ then $f(s_1) \neq f(s_2)$

1) Definition of 1-1 is
if $f(x) = f(y) \Rightarrow x = y$

So yes

2) Not necessary take $f(x) = x^2$ $f : \mathbb{R} \to \mathbb{R}^+$
which is NOT one to one.

So No yet if you take any $x_1, x_2 \in \mathbb{R}$
s.t $x_1 = x_2$ then of
course $x_1^2 = x_2^2$!

3) This is the contrapositive of 1)

So yes.

(b) (12%) Let $\mathbb{P}$ denote the set of positive integers. Let $f : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be given by $f(m,n) = 3^m 7^n$. Show that this function is one to one but not onto

1-1 $3^{m_1} 7^{n_1} = 3^{m_2} \cdot 7^{n_2}$

$3^{m_1 - m_2} \cdot 7^{n_1 - n_2} = 1$

1-1 ?

Let $f(m_1, n_1) = f(m_2, n_2)$

for $m_1, n_1, m_2, n_2 \in \mathbb{P}$

then $3^{m_1} 7^{n_1} \stackrel{\checkmark}{=} 3^{m_2} \cdot 7^{n_2}$

$\Rightarrow 3^{m_1 - m_2} \cdot 7^{n_1 - n_2} = 1$

Since 3 & 7 have no common divisors this necessarily implies that

$m_1 - m_2 = 0$ & $n_1 - n_2 = 0$

$\Rightarrow m_1 = m_2$ & $n_1 = n_2$

$\Rightarrow (m_1, n_1) = (m_2, n_2)$

Onto ?

No take any integer between 21 and 9×7, it won't have a pre-image

(c) (12%) Let $f : S \to T$ and $g : T \to U$ denote two one to one functions. Show that $gof : S \to U$ is one to one

let

$g\circ f\,(x_1) = g\circ f(x_2)$ any $x_1, x_2 \in S$

then this is the same as

$g\,[\overbrace{\boxed{f(x_1)}}^{y_1}] = g\,[\overbrace{\boxed{f(x_2)}}^{y_2}]$ $\quad f(x_1)\ f(x_2) \in T$

$\Rightarrow f(x_1) = f(x_2)$ since g is 1-1

$\Rightarrow x_1 = x_2$ since f is 1-1

So we just proved that gof is 1-1 by applying the definition