Lebanese American University **Fall 2010**

Byblos

Discrete Structure I **Date:** 21/12/2010

Test #2 **Duration:** 1h 30

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1. The following equalities are true or not? Either prove it or give a counter-example.
   (a) $A \cap (B - C) = (A \cap B) - (A \cap C)$
   (b) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$

2. We consider, from $\mathbb{R}$ to $\mathbb{R}$, the functions $f(x) = x - 1$ and $g(x) = x^2 + 1$.
   (a) Find the functions $g \circ f$, $f \circ g$, $f + g$ and fg.
   (b) Which functions in the preceding question are one-to-one? Explain.

3. Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 3, 4, 5\}$. Find the cardinal of the following sets:
$$\mathcal{P}(A),\ \ \mathcal{P}(B),\ \ \mathcal{P}(A) \cup \mathcal{P}(B),\ \ \mathcal{P}(A \times B),\ \ \mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B)),$$
$$\mathcal{P}(A \times B) \cup \{\{(1,1)\}\} \text{ and } \mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B)) \cup \{(\{1\}, \{1\})\}$$

4. Give an example of a function from $\mathbb{Z}$ to $\mathbb{Z}$ that is:
   (a) one-to-one but not onto.
   (b) onto but not one-to-one.
   (c) both one-to-one and onto (but different from the identity function).
   (d) neither one-to-one nor onto.

5. We denote by $\{0, 1\}^{\mathbb{N}}$ the set of all infinite strings over the alphabet $\Sigma = \{0, 1\}$.
   (a) Prove that $\{0, 1\}^{\mathbb{N}}$ is uncountable.
   (b) Find a bijection between $\mathcal{P}(\mathbb{N})$ and $\{0, 1\}^{\mathbb{N}}$.
   (c) Deduce that $\mathcal{P}(\mathbb{N})$ is uncountable.

MARKS : 1. [20] **2.** [25] **3.** [20] **4.** [25] **5.** [20]

Lebanese American University **Fall 2011**

Byblos

Discrete Structure I **Date:** 08/12/2011

Test #2 **Duration:** 1h 30

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1. Prove that if p is a prime number then $\sqrt[3]{p}$ and $\sqrt[3]{p^2}$ are irrational numbers.

2. The following equalities are true or not? Either prove it or give a counter-example.

   (a) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

   (b) $(A - B) \oplus C = (A \oplus C) - (B \oplus C)$

3. Prove, by induction, the following:

   (a) $n^3 - n$ is always divisible by 6 for all $n \in \mathbb{N}$.

   (b) $3^{6n} - 2^{6n}$ is always divisible by 35 for all $n \in \mathbb{N}^*$.

   (c) The sum of the first n positive odd numbers is n^2.

   (d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$.

4. Suppose that $A = \{1, 2, \{3, 5\}\}$ and $B = \{1, \{3\}, 4, \{6, 7\}\}$. Find the cardinal of the following sets:

   A, B, $A \cap B$, $A \cup B$, $\mathcal{P}(A)$, $\mathcal{P}(B)$, $\mathcal{P}(A \cup B)$, $\mathcal{P}(A) \cup \mathcal{P}(B)$, $\mathcal{P}(A \times B)$, $\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))$, $\mathcal{P}(A \times B) \cup \{\{(\{3, 5\}, 1)\}\}$ and $\mathcal{P}(A \times B) \cup \{\{(\{3, 5\}, \{1\})\}\}$

5. Prove that if A, B and X are three sets then:

$$X \subset A \cup B \Longleftrightarrow (X - A) \cap (X - B) = \emptyset.$$

6. Prove by a counterexample that the following implication is false:

$$A \subset B \Longrightarrow A \oplus C \subset B \oplus C$$

MARKS : 1. [14] **2.** [16] **3.** [40] **4.** [24] **5.** [8] **6.** [8]

Lebanese American University **Fall 2012**

Byblos

Discrete Structure I **Date:** 17/12/2012
Test #2 **Duration:** 2h

Name: **ID:**

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1. (a) The following assertion is true or false? Explain.

   The quotient of two positive and distinct irrational numbers is irrational.

   (b) Prove that if p and q are two distinct prime numbers then $\sqrt{p/q}$ is irrational.

2. Let A, B and C be three sets. The following equalities are true or not? Either prove it or give a counter-example.

   (a) $(A \cap B) \oplus C = (A \oplus C) \cap (B \oplus C)$.

   (b) $(A - B)^c = A^c \cup (A \cap B)$.

3. Solve the following two independent questions:

   (a) Find a set with two elements, such that every element of the set is also a subset of it.

   (b) What is the bit string corresponding to the symmetric difference of two sets?

4. Prove, by induction, the following:

   (a) $n^2 < n!$ for all $n \geq 4$.

   (b) $2^{2n+1} + 1$ is divisible by 3 for all $n \in \mathbb{N}$.

   (c) $5^{n+1} + 2 \cdot 3^n + 1$ is divisible by 8 for all $n \in \mathbb{N}$.

   (d) $4 \sum_{i=0}^{n} i3^i = (2n-1)3^{n+1} + 3$ for all $n \in \mathbb{N}$.

5. We consider the following function, called *Collatz function*, defined from $\mathbb{N}$ to $\mathbb{N}$ by:

$$f(n) = \begin{cases} 3n+1, & n \text{ is odd,} \\ \\ n/2, & n \text{ is even.} \end{cases}$$

   (a) Find $f(3)$, $f(20)$ and $f^{-1}(10)$.

   (b) Is f one-to-one? Explain.

   (c) Is f onto? Explain.

   (d) For $k \in \mathbb{N}^*$, we denote by f^k the function $\underbrace{f \circ f \circ \cdots f \circ f}_{k}$.

$\longrightarrow$

i. Find $f^5(5)$, $f^3(8)$ and $f^8(6)$.
ii. Find k such that $f^k(26) = 1$.
iii. What do you remark? Explain.

6. We consider the following program where the inputs are m and n (in $\mathbb{N}$) and the output is m.

```
while n > 0 do
   begin n := n - 1;
          m := m + 3;
   end;
```

(a) Find the output of the program that corresponds to each of the following inputs (m, n): (4,3), (3,2) and (2,4).

(b) Which function computes the program?

(c) Write a correctness proof for this program.

MARKS : 1. [12] **2.** [15] **3.** [12] **4.** [28] **5.** [25] **6.** [18]

Lebanese American University **Fall 2013**

Byblos

Discrete Structure I **Date:** 18/12/2013
Test #2 **Duration:** 2h

Name: **ID:**

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1. Prove that the following numbers are irrational:

   (a) $\sqrt[3]{3}$ (b) $\sqrt{2}+\sqrt[3]{3}$ (c) $\log_5 2$

2. Let A, B and C be three sets. The following assertions are true or not? Either prove it or give a counter-example.

   (a) $(A \cup B) - (C - A) = A \cup (B - C)$.
   (b) $(A \oplus B) \cup C = (A \cup C) \oplus (B \cup C)$.
   (c) $(B \cap C) \subset A \implies (A - B) \cap (A - C) = \emptyset$.

3. Prove, by induction, the following:

   (a) $7^n - 6n - 1$ is divisible by 36 for all $n \in \mathbb{N}^*$.
   (b) $n(n^2+5)$ is divisible by 6 for all $n \in \mathbb{N}$.
   (c) $\sqrt{n} \leq \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$

4. We denote by $\mathcal{E}$ the set of all even natural numbers. Give the formula of a function from $\mathbb{Z}$ to $\mathcal{E}$ that is:

   (a) onto but not one-to-one.
   (b) one-to-one but not onto.
   (c) both one-to-one and onto.
   (d) neither one-to-one nor onto.

5. We consider the following program where the inputs are m and n (in $\mathbb{N}$) and the output is m.

```
while n > 0 do
   begin n := n - 1;
         m := m * m;
   end;
```

   (a) Find the output of the program that corresponds to each of the following inputs (m, n): (2,3), (3,2) and (2,4).
   (b) Which function computes the program?
   (c) Write a correctness proof for this program.

6. (a) Prove that the set of all irrational numbers is uncountable.
   (b) Give an example of a countable set A that contains only irrational numbers.
   (c) Find a bijection between your set A and $\mathbb{N}$.

MARKS : 1. [18] **2.** [18] **3.** [24] **4.** [16] **5.** [16] **6.** [18]

Lebanese American University **Spring 2014**

Byblos

Discrete Structure I **Date:** 19/05/2014
Test #2 **Duration:** 1h 30

Name: **ID:**

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1. The set $A = \{x \in \mathbb{R} : \ln(x) \in \mathbb{Z}\}$ is countable or not? Interpret you answer.

2. (a) Prove, by contradiction, that $\sqrt[3]{5}$ is irrational.
   (b) Let n be a positive integer. Prove that:
   $$n - 1 \text{ is divisible by } 3 \implies \sqrt[3]{5^n} \text{ is irrational.}$$

3. Let A, B and C be three sets. The following assertions are true or not? Either prove it or give a counter-example.
   (a) $A - (B \oplus C) = (A - B) - C$
   (b) $B \cap (A - C) = A \cap (B - (A^c \cup C))$

4. Prove by induction that:
   (a) $7^{2n} - 48n - 1$ is divisible by 2304 for all $n \in \mathbb{N}$.
   (b) $2^{n+1} \geq n^2 + 2$ for all $n \in \mathbb{N}$.

5. Let a, b and c in $\mathbb{Z}$.
   (a) Prove, by contradiction, that:
   $$a^2 + b^2 \text{ is divisible by } 4 \implies a \text{ or } b \text{ is even}$$
   (b) Deduce that:
   $$a^2 + b^2 = c^2 \implies a \text{ or } b \text{ is even.}$$

6. We consider the following program where the inputs are m and n (in $\mathbb{N}$) and the output is m.

```
while n > 0 do
    begin n := n - 1;
          m := m + m;
    end;
```

   (a) Find the output of the program that corresponds to the inputs: (3,2) and (2,3).
   (b) Which function computes the program?
   (c) Write a correctness proof for this program.

7. We denote by $\mathcal{E}$ the set of all even <u>integers</u> and by $\mathcal{O}$ the set of all odd <u>natural numbers</u>. Give the formula of a function from $\mathcal{E}$ to $\mathcal{O}$ that is:

   (a) onto but not one-to-one.

   (b) one-to-one but not onto.

   (c) both one-to-one and onto.

   (d) neither one-to-one nor onto.

MARKS : 1. [7] **2.** [10] **3.** [20] **4.** [20] **5.** [12] **6.** [16] **7.** [25]