**Lebanese American University** **Fall 2009**

**Byblos**

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**Discrete Structure I** **Date:** 27/01/2009

**Final Exam** **Duration:** 2h

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1. We consider three sets, $A$, $B$ and $C$. The equality

$$(A - B)\Delta\, C = (A\,\Delta\, C) - (B\,\Delta\, C)$$

is true or not? Either prove it or give a counter-example.

2. Prove, by induction, that $3^{6n} - 2^{6n}$ is always divisible by 35 for all $n \in \mathbb{N}^*$.

3. (a) What is the number of six digit numbers?
   (b) What is the number of six digit numbers that have no two consecutive digits equal?
   (c) What is the number of six digit numbers that have at least one pair of consecutive digits equal?

4. Verify, without using a truth table, if $\Gamma \models \varphi$ holds or not:

$$\Gamma = \{r \longrightarrow (p \vee s)\,;\, \neg s \longrightarrow (q \vee \neg r)\,;\, r\ \ ;\neg s\} \text{ and } \varphi = p \wedge \neg q$$

5. Verify, in the following cases, if the binary relation $R$ defined over $\mathbb{R}$ is reflexive, symmetric or transitive. If $R$ is an equivalence relation then find the set of classes $\pi_R$.
   (a) $x\,R\,y$ iff $y = |x|$.
   (b) $x\,R\,y$ iff $\sin(x - y) = 0$.
   (c) $x\,R\,y$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.
   (Note that the $\lfloor x \rfloor$ is the largest integer smaller than $x$).

6. Use the laws of Boolean algebra to simplify the following expressions:
   (a) $(x \vee y) \wedge (-x \vee y)$
   (b) $(x \wedge y) \vee (x \wedge -y) \vee (-x \wedge y)$
   (c) $(x \wedge -y \wedge z) \vee (x \wedge y \wedge (-x \vee z)) \vee (y \wedge z \wedge (-y \vee x))$

$\longrightarrow$

7. Solve, using pigeonhole principle, **one** of the following two independent questions:

   (a) We consider in the plane five points with integer coordinates. Prove that at least one pair of them have a midpoint with integer coordinates.

   (b) Five points are chosen from inside an equilateral triangle of side 2. Prove that two points must be within a distance of 1 of each other.

8. *Bonus question.* Over the set $\mathbb{N}^*$ we define the following binary relation:

$$a\,R\,b \text{ iff } ab \text{ is a square.}$$

   (a) Prove that $R$ is an equivalence relation.

   (b) Find the that set of classes $\pi_R$.

**MARKS : 1.** [10] **2.** [15] **3.** [15] **4.** [15] **5.** [20] **6.** [15] **7.** [10]

**Bonus question** [10]

**Lebanese American University** **Fall 2010**

**Byblos**

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**Discrete Structure I** **Date:** 27/01/2011

**Final Exam** **Duration:** 2h

**Name:** **ID:**

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## Part I: Problems (70 pts)

1. [**10 pts**] Find a resolution proof and a formal proof for the following propositional logic formula:

$$\{A \longrightarrow B\,;\, \neg A \longrightarrow C\,;\, C \longrightarrow D\} \vdash \neg B \longrightarrow D$$

2. [**15 pts**] Prove, by induction, the following:

   (a) $3^{6n} - 2^{6n}$ is always divisible by 35 for all $n \in \mathbb{N}^*$.

   (b) $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$.

3. [**10 pts**] Solve using pigeonhole principle the following two independent questions:

   (a) A computer network consists of eight computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same numbers of other computers.

   (b) 1335 people are seated in a row of 2001 chairs. Prove that there are 3 consecutive non-empty chairs.

4. [**10 pts**] We consider the following program (input is $n$ and output is $r$):

```
r:=0;
if n>0 then
   begin r:=1;
         i:=1;
         while i<n do
           begin i:=i+1;
                 r:=r+3*i*i-3*i+1;
           end;
   end;
```

(a) Find the output of the program for $n = 1$, $n = 2$ and $n = 3$.

(b) Which function computes the program?

5. [**15 pts**] Verify, in the following cases, if the binary relation $R$ defined over the set $X$ is an equivalence relation or not. If $R$ is an equivalence relation then find the set of classes $\pi_R$.

(a) $X = \mathbb{R}$ and $x\,R\,y$ iff $|x| + |y| = |x + y|$.

(b) $X = \mathbb{R}^*$ and $x\,R\,y$ iff $xy > 0$.

(c) $X = \mathbb{R}$ and $x\,R\,y$ iff $x^2 - xy + 2x - 2y = 0$.

6. [**10 pts**] Find for the following two ordered sets the maximal, minimal, greatest and least elements (if any):

   (a) $X = \{2, 3, 6, 12, 18, 24\}$ and the order is $x\,R\,y$ iff $x$ divides $y$.

   (b) $X = \{\{2\}, \{2, 3\}, \{2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ and the order is $x\,R\,y$ iff $x \subseteq y$.

**Lebanese American University** **Fall 2011**

**Byblos**

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**Discrete Structure I** **Date:** 25/01/2012

**Final Exam** **Duration:** 2h 15

**Name:** **ID:**

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## Part I: Problems (70 pts)

1. [**10 pts**] Find a resolution proof and a formal proof for the following propositional logic formula:

$$\{C \longrightarrow (A \vee D)\,;\ \neg D \longrightarrow (B \vee \neg C)\,;\ C\,;\ \neg D\} \vdash \neg(A \longrightarrow \neg B)$$

(*Hint*: For the formal proof, assume $A \longrightarrow \neg B$ and prove a contradiction)

2. [**15 pts**] Prove, by induction, the following:

(a) $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{N}$.

(b) $\displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$.

(c) $5^{2n+1} \cdot 2^{n+2} + 3^{n+2} \cdot 2^{2n+1}$ is divisible by 19 for all $n \in \mathbb{N}$.

3. [**10 pts**] Solve using pigeonhole principle the following two independent questions:

   (a) There are 25 students in a class. While doing a keyboarding test, each student made fewer than 12 mistakes. Show that at least 3 students made the same number of mistakes.

   (b) Five points are chosen from inside an equilateral triangle of side 2. Prove that two points must be within a distance of 1 of each other.

4. [**10 pts**] We consider the following program (input is $n$ and output is $r$):

```
if n > 0 then
   begin r := 0;
         i := 1;
         while i <= n do
           begin r := r + 2*i;
                 i := i + 1;
           end;
   end;
```

(a) Find the output of the program for $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

(b) Which function computes the program?

5. **[13 pts]** Verify, in the following cases, if the binary relation $R$ defined over the set $X$ is reflexive, symmetric and transitive. If $R$ is an equivalence relation then find the set of classes $\pi_R$.

   (a) $X = \mathbb{Z}$ and $x\,R\,y$ iff $x + y$ is even.

   (b) $X = \mathbb{Z}$ and $x\,R\,y$ iff $xy$ is even

6. [**12 pts**] Find for the following two ordered sets the maximal, minimal, greatest and least elements (if any):

   (a) $X = \{2, 6, 12, 18, 24, 36\}$ and the order is $x\,R\,y$ iff $x$ divides $y$.

   (b) $X = \{[0, 1], [-1, 2], [-1, 3], [2, 3], [-1, 5]\}$ and the order is $x\,R\,y$ iff $x \subseteq y$.

   (c) $X = \{2, 3, 4, 8, 9, 16\}$ and the order is $x\,R\,y$ iff $\exists\, n \in \mathbb{N}^*$ such that $y = x^n$.

**Lebanese American University** **Fall 2012**

**Byblos**

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**Discrete Structure I** **Date:** 25/01/2013
**Final Exam** **Duration:** 2h 15

**Name:** **ID:**

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## Part I: Problems (68 pts)

1. [**8 pts**] Find a resolution proof and a formal proof for the following propositional logic formula:

$$\{\neg A \longrightarrow (\neg C \vee D)\,;\, C \longrightarrow (D \vee B)\,;\, C \longrightarrow \neg D\,;\, C\} \vdash A \wedge B$$

2. [**10 pts**] Find $A^n$ where $A$ is the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

3. [**12 pts**] Prove, by induction, the following:

(a) $2^{6n} + 3^{2n-2}$ is divisible by 5 for all $n \in \mathbb{N}^*$.

(b) $\displaystyle\sum_{k=2}^{n} \binom{k}{2} = \binom{n+1}{3}$ for all $n \geq 2$.

4. **[8 pts]**

(a) Prove that if three natural numbers $x$, $y$ and $z$ have same remainder on dividing by 3, then $x+y+z$ is divisible by 3.

(b) Prove that among any seven different positive integers, it is possible to choose three such that their sum is divisible by 3.

5. [**10 pts**] We consider the following program where the inputs are $m$ and $n$ (both in $\mathbb{N}$) and the output is $m$.

```
while n > 0 do
  begin n := n - 1;
        m := m + m;
  end;
```

(a) Find the output of the program that corresponds to each of the following inputs $(m, n)$: (4,3), (3,2) and (2,4).

(b) Which function computes the program?

(c) Write a correctness proof for this program.

6. **[10 pts]** Verify, in the following cases, if the binary relation $R$ defined over the set $X$ is an equivalence relation or not. If $R$ is an equivalence relation then find the set of classes $\pi_R$.

   (a) $X = \mathbb{Z}$ and $x\,R\,y$ iff $x^2 - y^2 = x - y$.

   (b) $X$ is the set of prime numbers greater than 2 and $x\,R\,y$ iff $\dfrac{x+y}{2}$ is prime.

7. [**10 pts**] Let $X$ be a nonempty set of positive natural numbers and let $R$ be the binary relation defined by: $x\,R\,y$ iff $\exists\, n \in \mathbb{N}^*$ such that $y = x^n$.

   (a) Prove that $R$ is an order relation.

   (b) Find the maximal, minimal, greatest and least elements of $(X, R)$ if:

      i. $X = \{3, 27, 243, 729\}$.

      ii. $X = \{2, 4, 8, 64, 192, 512, 4096\}$

**Lebanese American University** **Spring 2014**

**Byblos**

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**Discrete Structure I** **Date:** 09/06/2014

**Final Exam** **Duration:** 2h

**Name:** **ID:**

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1. Let $A$, $B$ and $C$ be three sets. The following assertion is true or not? Either prove it or give a counter-example.
$$(A - B) \cup (C - B) = (A \cup C) - B$$

2. Find a resolution and a formal proof for the following:
   (a) $\{\neg B\} \vdash ((A \longrightarrow B) \longrightarrow B) \longrightarrow A$
   (b) $\vdash (B \longrightarrow A) \vee (A \longrightarrow C)$

3. Find $A^n$ and $B^n$ where $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$.

4. Prove, by contradiction, the following two (independent) statements:
   (a) If $a, b$ and $c$ are three natural numbers such that $a^2 + b^2 = c^2$, then $abc$ is even.
   (b) There does not exist a natural number $n$ such that $4n + 3$ is a perfect square.

5. Solve, using pigeonhole principle, the following two (independent) questions:
   (a) Prove that if you pick five numbers from the integers 1 to 8, then two of them must add up to nine.
   (b) Choose 13 points on a square with side length 1. Show that there must have four points to form a quadrilateral whose area is less than or equal to 1/4.

6. Prove by induction that:
   (a) $5^{3n+7} + 6^{2n-1} + 4$ is divisible by 5 for all $n \in \mathbb{N}^*$.
   (b) $5^n + 9 < 6^n$ for all $n \geq 2$.

7. Use the laws of Boolean algebra to simplify the following expressions:
   (a) $(-(x \wedge y)) \wedge (-x \vee y) \wedge (-y \vee y)$
   (b) $[-x \wedge (x \vee y)] \vee [(x \vee y) \wedge (x \vee -y)]$

8. Let $R$ be the binary relation defined on $\mathbb{R}$ by $x\,R\,y$ iff $2x^2 - 3xy + y^2 = 0$. Check if $R$ is reflexive, symmetric, antisymmetric or transitive.

9. Find, for the following order, the maximal, minimal, greatest and least elements:
$$X = \{\{0\}, \{0, 1\}, \{0, 2\}, [-2, 0], [1, 3], [-2, 3]\} \text{ and the order is } x\,R\,y \text{ iff } x \subseteq y.$$

**MARKS : 1.** [8] **2.** [16] **3.** [12] **4.** [12] **5.** [12] **6.** [16] **7.** [10] **8.** [12] **9.** [12]