Discrete Structures I
Fall 2013
January 2013
Exam III

Name: Solutions

1. (8%) Use the Euclidean algorithm to find the gcd(27,72). Also write the gcd as a linear combination of the 2 numbers, showing details of your work.

$72 = 27(2) + 18$.

$27 = 18(1) + 9 \longleftrightarrow \gcd = 9$.

$18 = 9(2) + 0$

$9 = 27(3) - 72$

2. (10%) Given three integers $a, b$, and $m$, and if $(m, a) = 1$, show that if $m \mid ab$, then $m \mid b$?

If $m$ and $a$ have no common prime factors .. and $ab = mq \Rightarrow$

$$mq = ab = \underbrace{(p_1^{\alpha_1} \cdots p_k^{\alpha_k})}_{a} \cdot \underbrace{(q_1^{\beta_1} \cdots q_r^{\beta_r})}_{b}$$

But the prime divisors of $m$ don't include any $p_i$ $\Rightarrow$ they must include some of the $q_i$'s $\Rightarrow$

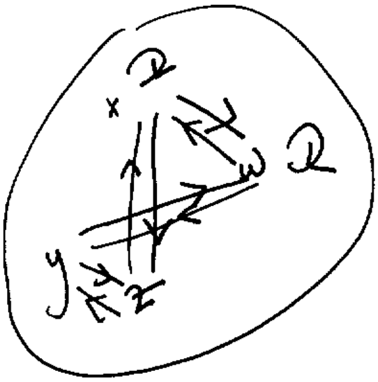
$m \mid b$

3. (10%)Prove that if $a$ divides $b$, then $a^2$ divides $b^2$.

$b = aq \Rightarrow b^2 = q^2a^2 \Rightarrow a^2 \mid b^2.$

4. (10%)Construct a graph $G$ on the set of vertices $\{x, y, z, w\}$whose matrix is $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

(a) Find all paths from $x$ to $y$. Mention their length




$x\ z\ y$

$x\ w\ y$

$x\ w..w\ y$

$x..\ x\ z\ y$

(b) Find all paths of length 2 using matrices.

$$M^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 1 & 2 & 3 \end{bmatrix}$$

5. (7%)Given the following partition on the set $N : \{n \in N : n = 7k + p\}$, where $p = 0, 1, 2, 3, 4, 5, 6$.

(a) Find an equivalence relation $\sim$ on the set $N$ that partitions $N$ into the sets mentioned in the partitions above

$mRn$ iff $m \equiv n \pmod 7$

6. (10%) Consider the relation $R$ on $A=\{1,2,3,4,5,6\}$, $aRb$ if and only if $a+b \leq 8$. State all properties of $R$.

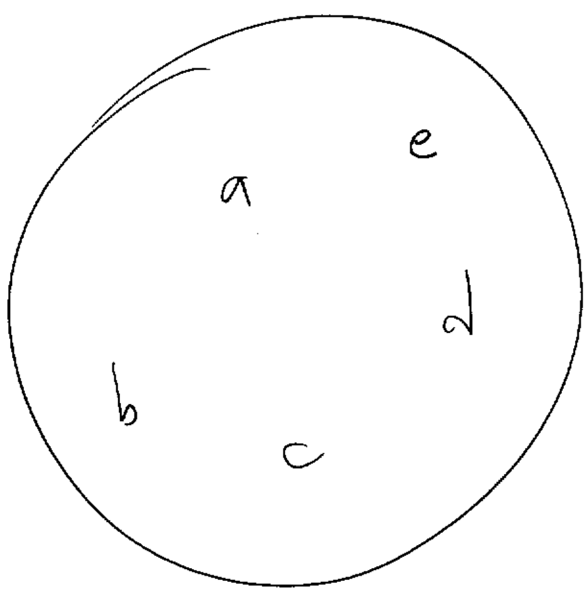
1) not (R) since $6 \not R 6$.

2) not (AR) since $1 R 1$

3) (S) since $a+b=b+a$

4) not (T) since $5R3$, $3R5$, $5 \not R 5$

7. (8%) Find, if possible, using a digraph, a relation on the set $A = \{a,b,c,d,e\}$ that is (AR),(S) and (T). Comment on your construction.




Must be $\phi$

since if not

$\Rightarrow$

$a \to b$

$\Rightarrow$

$a \rightleftarrows b$ since (S)

$\Rightarrow$ $a \rightleftarrows b$ since (T)

$\Rightarrow$ no more (AR) !!

8. (10%) Let $A = \mathbb{Z}$, and $R$ be the relation on $A$ given by $aRb$ if and only if $a^2 \equiv b^2 (\mathrm{mod}\, 5)$.

(a) Show (in a very efficient way) that the relation $R$ is an equivalence relation.

$aRb \Leftrightarrow f(a) = f(b)$ where $a^2 = b^2 \pmod 5$

(b) Find all equivalence classes determined by $R$.

| $m \bmod 5$ | $m^2 \bmod 5$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 4 |
| 4 | 1 |

$[0] = \{5k\}$

$[1] = \{5k+1, 5k+4\}$

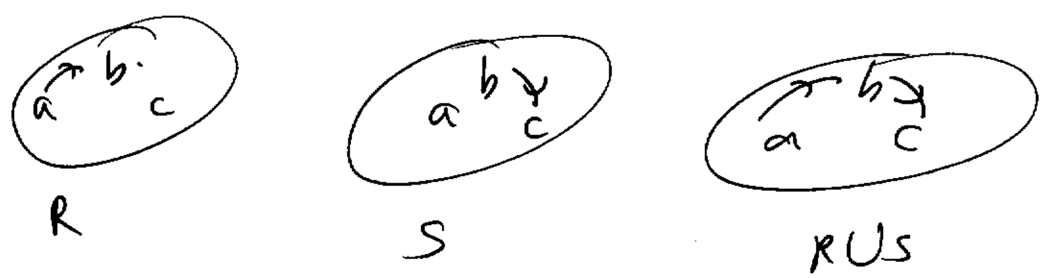
$[2] = \{5k+2, 5k+3\}$

8 of [0], [1], [4]

9. (10%) Answer the following. In case the answer is yes, prove your statement, and in case it is no, find a real counterexample.

(a) If $R$ and $S$ are two transitive relations, is $R \cup S$ also transitive?

No




3 of no ex

$R \cup S$ not (T)

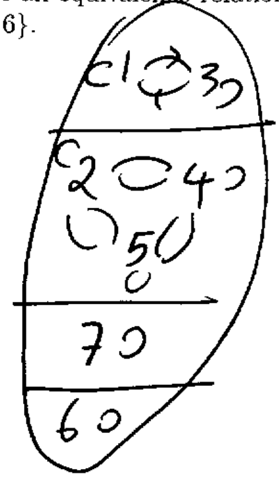
(b) If $R$ and $S$ are two symmetric relations, is $R \cup S$ also symmetric? Yes

If $a(R \cup S)b \Rightarrow$ either $aRb$ or $aSb \Rightarrow$ either $bRa$ or $bSa \Rightarrow b(R \cup S)a$

$\Rightarrow a(R \cup S)b \Rightarrow b(R \cup S)a \quad \forall a, b \Rightarrow$

$R \cup S$ must be (S)

10. (7%) Draw the digraph of an equivalence relation on $A = \{1,2,3,4,5,6,7\}$ whose equivalence classes are $\{1,3\}, \{2,4,5,\}, \{7\}, \{6\}$.




5 ½ no arrows shown

11. (10%) Let $A = \{a,b,c,d\}$ and consider the matrix $M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(a) Find all the paths of length 2 from vertex $a$ to each of the other vertices: We need the 1º row of $M^2$ only!!

$$M^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$a \to a$ (2 paths) $\quad a \to b \quad a \to c$

(b) Find $R(a)$ and $R(\{b,c,d\})$

$R(a) = \{a, b\}$

$R(b, c, d) = \{a, c, b, d\} = A.$