AMERICAN UNIVERSITY OF BEIRUT

~~(MATHEMATICS)~~ 218

FALL SEMESTER 2011-2012

EXAM I

Time: 60 minutes

Name:..

ID:...

Section:..

Instructor:..

Answer the following sets of questions; the back of pages may be used as scratch.

NO QUESTIONS ARE ALLOWED.

GRADE........................./100

1- Find the values of m for which the vector $\begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}$ is a linear combination of

$\begin{pmatrix} m \\ m \\ 0 \end{pmatrix}$, $\begin{pmatrix} m \\ 0 \\ m \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ m-1 \end{pmatrix}$.

$$c_1 \begin{bmatrix} m \\ m \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} m \\ 0 \\ m \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ m-1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} m & m & 3 & -4 \\ m & 0 & 1 & 2 \\ 0 & m & m-1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} m & m & 3 & -4 \\ 0 & -m & -2 & 6 \\ 0 & m & m-1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|c} m & m & 3 & -4 \\ 0 & -m & -2 & 6 \\ 0 & 0 & m-1 & 6 \end{array}\right]$$

for $m \neq 3$ then $m - 3 \neq 0$

so the system has a unique solution.

$$c_1 \begin{bmatrix} m \\ m \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} m \\ 0 \\ m \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ m-1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} m & m & 3 & -4 \\ m & 0 & 1 & 2 \\ 0 & m & m-1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} m & m & 3 & -4 \\ 0 & -m & -2 & 6 \\ 0 & m & m-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} m & m & 3 & -4 \\ 0 & -m & -2 & 6 \\ 0 & 0 & m-3 & 6 \end{array}\right]$$

$\cancel{R_3: mx_3 - 3 \neq 0 \quad x_3 = \frac{3}{m}}$

$\cancel{R_2: -m\ 2}$

$m - 3 \neq 0$

$m \neq 3$.

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

2- Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

i- Let $A = aI + bJ$ where I is the identity matrix. Find the values of a and b.

ii- Use the expression of A found in part i- to compute A^2 as a function of I and J.

iii- Show that $A^{-1} = \frac{(7I-A)}{10}$

i. $A = aI + bJ$

$$\begin{bmatrix}3&1&1\\1&3&1\\1&1&3\end{bmatrix} = a\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + b\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$$\left(\frac{7I-A}{10}\right) \cdot A$$

fa $a = 2$ and $b = 1$

$A = 2I + bI = \frac{7A - A^2}{10}$

ii- $A = 2I + J$

$A^2 = (2I+J)^2 = 4I^2 + 4IJ + J^2 = \frac{7(2I+J) - 4I - 7J}{10}$

$A^2 = 4I^2 + 4IJ + J^2$

$A^2 = 4I + 4J + J^2 = 4I + 7J$ $= I$.

iii-

$$\left[\begin{array}{ccc|ccc}3&1&1&1&0&0\\1&3&1&0&1&0\\1&1&3&0&0&1\end{array}\right]$$

$3R_2 - R_1 \to R_2$
$3R_3 - R_1 \to R_3$

$$\frac{(7I-A)}{10} = \frac{\left(\begin{bmatrix}7&0&0\\0&7&0\\0&0&7\end{bmatrix} - \begin{bmatrix}3&1&1\\1&3&1\\1&1&3\end{bmatrix}\right)}{10}$$

$$= \frac{1}{10}\begin{bmatrix}4&-1&-1\\-1&4&-1\\-1&-1&4\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}3&1&1&1&0&0\\0&8&2&0&3&0\\0&2&8&0&0&3\end{array}\right]$$

$8R_1 - R_2 \to R_1$
$4R_3 - R_2 \to R$

$$\left[\begin{array}{ccc|ccc}24&0&6&2&-3&0\\0&8&2&0&3&0\\0&0&22&-1&0&12\end{array}\right]$$

3- Let A be a 3×3 matrix. If $(A-2I)^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ then find A.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & -2 & 1 & 0 \\ 0 & -2 & 1 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} 3R_1 + R_2 \to \\ \frac{3}{2}R_3 + R_2 \to \end{array}$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 4 & 0 \\ 0 & -3 & 1 & -2 & 1 & 0 \\ 0 & 0 & \frac{5}{2} & -5 & 1 & \frac{3}{2} \end{array}\right] \quad \begin{array}{l} -\frac{5}{2}R_1 + R_3 \to R_1 \\ -\frac{5}{2}R_2 + R_1 \to R_2 \end{array} \quad \Bigg[$$

4- Suppose A is an $n \times n$ symmetric matrix. Show that $2A^2 - 3A + A^tA$ is also symmetric.

$2A^2 - 3A + A^tA = 2A^2 - 3A + A^2$

$= 3A^2 - 3A$

Since A is symmetric then A^t is also symmetric

so $2A^2 - 3A$ is symmetric. $A^t = A$

$(2A^2 - 3A + A^tA)^t = (2A^2)^t + (-3A)^t + (A^tA)^t$

~~$= 2A^2 - 3A + AA^t$~~

$=$

$= 2(AA)^t - 3(A)^t + (A^tA)^t$ ~~$(AB)^t = B^tA^t$~~

$= 2A^tA^t - 3A^t + A^tA = 2A^2 - 3A + A^tA$

5- Suppose $det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2.$

i- Find $det(3A^{-1})$ where $A = \begin{pmatrix} a+c & 3c \\ b+d & 3d \end{pmatrix}$. $\rightarrow \begin{bmatrix} a+c & b+d \\ 3c & 3d \end{bmatrix}$

ii- Find $det\begin{pmatrix} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d-2 & 2 & 2b \end{pmatrix}$.

i. $det(3A^{-1}) = |3A^{-1}| = 3^2|A^{-1}| = \frac{3^2}{|A|}$.

$det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1+R_2 \to R_1} = det\begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} \xrightarrow{R_2\times 3 \to R_2}$

$= \frac{1}{3} det\begin{bmatrix} a+c & b+d \\ 3c & 3d \end{bmatrix} = \frac{1}{3} det\begin{bmatrix} a+c & 3c \\ b+d & 3d \end{bmatrix} = \frac{1}{3} det(A).$

hence $det(A) = 3\ det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 6.$

so $det(3A^{-1}) = \frac{3^2}{|A|} = \frac{9}{6} = \frac{3}{2}$

ii. $det\begin{bmatrix} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d-2 & 2 & 2b \end{bmatrix} \overset{row 1}{=} 2\ det\begin{bmatrix} -1 & 2a \\ 2 & 2b \end{bmatrix} + 2\ det\begin{bmatrix} c+1 & 2a \\ d-2 & 2b \end{bmatrix}$

$= 2(-2b-4a) + 2((c+1)2b - 2a(d-2))$

$= -4b - 8a + 2(2cb + 2b - 2ad + 4a)$

$= -\cancel{4b} - \cancel{8a} + 4cb + \cancel{4b} - 4ad + \cancel{8a}$

$= 4cb - 4ad$

$= -4(ad - bc)$

$= -4\ det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -8.$

6- Suppose u, v and w are linearly independent. Show that $u+v-2w$, $u-v-w$ and $u+w$ are linearly independent.

$c_1(u+v-2w) + c_2(u-v-w) + c_3(v+w) = 0$

$c_1u + c_1v - 2c_1w + c_2w - c_2v - c_2w + c_3v + c_3w = 0$

$u(c_1+c_2+c_3) + v(c_1-c_2) + w(-2c_1-c_2+c_3) = 0$

but u, v, w are l.i. then the above eq has only the trivial sol.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & -1 & +1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \quad \begin{array}{l} 2R_3 + R_2 \\ \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \quad \begin{array}{l} R_3: c_3 = 0 \\ R_2: -2c_2 - c_3 = 0 \\ \quad c_2 = 0 \\ R_1: c_1 = 0 \end{array}$$

then they are l.i.

7- Let A be the 3×3 diagonal matrix $A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$. Find all values of a for which

$A^3 = A$

$a^3 = a$

$a^2 - a = 0$

$a(a^2-1) = 0$

$a(a-1)(a+1) = 0$

$a=0 \quad a=1 \quad a=-1$

$A^3 = A\,A\,A$

$= \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} A$

$= \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 0 & 0 \\ 0 & a^3 & 0 \\ 0 & 0 & a^3 \end{bmatrix}$

$\begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

$a^3 = a$

hence $\boxed{a=0}$ and $a=1$

$\boxed{a=1}$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ k & i & & \end{bmatrix}$$

8- Let A be the 4×4 matrix such the sum of the entries of each row is 0 and let $x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

What is Ax? Deduce $det A$.

$Ax = A_1 x_1 + A_2 x_2 + A_3 x_3 + A_4 x_4$

$= A_1 + A_2 + A_3 + A_4$

$= 0$

Since $Ax = 0$ for any $x = \begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix}$ hence A is not invertible $|A| = 0$

$R_1 - R_2 \to R$
$R_2 - R_3 \to R_2$

$R_8 - R_9 \to R_8$
$R_9 - R_{10} \to R_9$

- **Bonus Problem**

Let A be the 10×10 matrix $A = \begin{pmatrix} a+b & a & a & \dots & a \\ a & a+b & a & \dots & a \\ a & a & a+b & \dots & a \\ \vdots & & & & \vdots \\ a & a & \dots & a & a+b \end{pmatrix}$.

Show that $det A = b^9(10a+b)$

$R_2 - R_1 \to R_2$
$R_3 - R_1 \to R_3$

$R_{10} - R_1 \to R_{10}$

$$\begin{bmatrix} a+b & a & a & \dots & a \\ -b & b & 0 & \dots & 0 \\ -b & 0 & b & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ -b & 0 & \dots & 0 & b \end{bmatrix}$$

$det A = 10ab^9 + b^{10}$

R_{10}