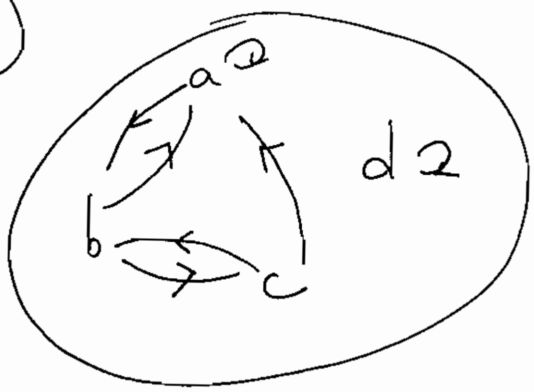
Solutions:

Discrete Structures I
Exam II
Fall 08

1. Let $A = \{a, b, c, d\}$ and $M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

6%

(a) Draw the digraph associated with the matrix.



(b) Find all the paths of length 2 from vertex a to each of the other vertices.

4%

a a a
a b a
a b c
a a b

(c) Evaluate M^2. What is the interpretation of this matrix?

6%

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is the # of paths of length 2.

(d) Find the relation R corresponding to M.(Write it as a set of ordered pairs)

4%

$R = \{(a,a), (a,b), (b,a), (b,c), (c,a), (c,b), (d,d)\}$

(e) Find the reachability matrix of R.

6%

ϕ

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Is it true that if A and B are two 3×3 matrices satisfying $AB = 0$ then one of them must be the zero matrix?Explain..

11%

No !

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Consider the sequence $\{a_n\}$, where $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}, \quad n \in \mathbb{Z}^+, \quad \text{where } n \geq 3$$

(a) Find a_3, a_4 and a_5.

3%

$a_3 = a_2 + a_1 + a_0 = 3 + 2 + 1 = 6$

$a_4 = a_3 + a_2 + a_1 = 6 + 3 + 2 = \cancel{10}\ 11$

$a_5 = a_4 + a_3 + a_2 = 1\cancel{0} + 6 + 3 = 19$

13%

(b) Prove by mathematical induction that for all $n \in \mathbb{N}$, we have that $a_n \leq 3^n$

$a_n \leq 3^n$.

① $a_3 = 6 \leq 3^3$ ✓ Basic Step

② Assume $a_k \leq 3^k$ $k = 1, 2, \ldots, n$

Show $a_{n+1} \leq 3^{n+1}$

$a_{n+1} = a_n + a_{n-1} + a_{n-2}$

$\leq 3^n + 3^{n-1} + 3^{n-2}$ show $\leq 3^{n+1}$

$3^n + 3^{n-1} + 3^{n-2} = 3^{n-2}\left[3^2 + 3 + 1\right] = 13 \cdot 3^{n-2} \leq 3^3 \cdot 3^{n-2}$

$\Rightarrow \leq 3^{n+1}$ Done.

3%

4. Define the sequence $\{s_n\}$ by $s_n = 6s_{n-1} - 9s_{n-2}$, and $s_0 = 1$ and $s_1 = 6$. Find the general formula for s_n in terms of n.

$a = 6$ $b = -9$.

$x^2 - ax - b = 0$ characteristic equation.

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$ Double roots $x = 3$.

$s_n = c r^n + c n r^n$

$s_n = c_1 3^n + c_2 n 3^n$. To find c_1, c_2 use $s_0 = 1$, $s_1 = 6$.

$n = 0$ $s_0 = 1 = c_1 + 0 \Rightarrow \boxed{c_1 = 1}$

$n = 1$ $s_1 = 6 = 3^1 + c_2 3 \Rightarrow \boxed{c_2 = 1}$

$\therefore$ $\boxed{s_n = 3^n + n3^n}$ Verify!!

5. We consider the equivalence relation R on the set N of positive integers defined by nRm if and only if $|a| = |b|$. Find the equivalence classes determined by R.

13%

$[m] = \{ m, -m \}$ $\forall\ m \in \mathbb{N}$.

6. Consider the relation mRn if and only if $m^2 \equiv n^2 \pmod 5$

(a) Show that the relation is an equivalence relation (without using reflexivity, symmetry and transitivity.

7%

$g(m) = m^2 \% 5$.

This function is so that $g(m) = g(n)$ whenever mRn

7%

(b) Find the equivalence classes determined by R.

m	m^2
0	0
1	1
2	4
3	4
4	1

$[0] = \{ 5k$

$[1] = \{ 5k+1, 5k+4 \}$

$[2] = \{ 5k+2, 5k+3 \}$

(c) Consider the function : $f : Z \to Z$ defined on the Equivalence classes of R. given by $f(m) = m^2\%5 + 3$. Verify if f is well defined or not.

7%

Yes it is well defined because

If $mRn \Rightarrow f(m) = f(n)$.