Discrete Structures I
Exam 2 make up (May 9, 2012)

Name: Solutions

1. Let $A = \{1, \{1\}, \{1,2\}, \phi, \{\phi\}\}$. Answer the following by True or False:

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(a) $\{1\} \in A$ T

(b) $\phi \subset A$ T

(c) $\phi \in A$ T

(d) $\{\{1, \phi\}\} \subset A$ F

(e) $\{\{1,2\}\} \subset A$ T

(f) $\{1, 2, \{\phi\}\} \subset A$ F

(g) $\{\Phi\} \in A$ T

(h) $\{\Phi\} \subset A$ T

(i) $\{\{\Phi\}\} \subset A$ T

2. Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ that is 1-1 but not onto.

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$f(n) = 2n$

3. Show in details that if $f : A \to B$ and $g : B \to C$ are both onto, then so is ~~$f \circ g$~~ $g \circ f$ $: A \to C$

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Let $c \in C$. Find $a \in A$ s.t. ~~$f \circ g$~~ $g \circ f(a) = c$. $g[f(a)] = c$.

Since ~~f~~ g is onto, $\exists\ b \in B$ s.t. $g(b) = c$. And since f is onto, $\exists\ a \in f$ s.t. $f(a) = b$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

$\therefore$ ~~A~~ $g[f(a)] = c \Rightarrow g \circ f\ (a) = c$ $\therefore$ $g \circ f$: onto

4. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m,n) = m^2 + n$

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(a) Is f 1-1?

No $f(2,1) = 6(-2,1) = 5$

$\therefore$ f not 1-1.

(b) Is f onto? Yes. Let $a \in \mathbb{Z} \Rightarrow f(0,a) = a$

$\therefore$ f is onto.

5. For which numbers x and y is it true that $\lceil x+y \rceil = \lceil x \rceil + \lfloor y \rfloor$?Explain

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1) obviously if $x, y \in \mathbb{Z}$ this applies

2) If $x = n + d$ where $n = \lfloor x \rfloor$
$y = m + d'$ $m = \lfloor y \rfloor$

$\Rightarrow$ we need $0 \leq d + d' < 1$

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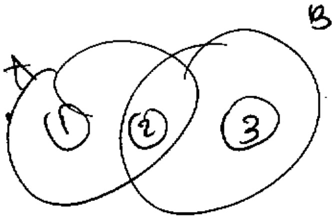
6. Under what conditions is $\overset{1,2,3}{A \cup B} = \overset{1,2,3}{A \cup (B - A)}$? Always



7. If $f : A \to B$ be 1-1.Show that $f(S \cup T) = f(S) \cup f(T)$, where S and T are subsets of A.

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Double inclusion:

1) $\overset{?}{\subseteq}$ Let $y \in f(S \cup T) \Rightarrow y = f(a)$ where $a \in S \cup T \Rightarrow a \in S$ or $a \in T \Rightarrow y \in f(S) \cup f(T)$.

2) $\overset{?}{\supseteq}$ Let $y \in f(S) \cup f(T) \Rightarrow y \in f(T)$ or $y \in f(T)$. $\Rightarrow y = f(a)$ where $a \in S$ or T.

Note: We don't need f to be 1-1 !!!

8. Show whether or not $f = O(f * g)$

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We need to assume $f, g \xrightarrow[x \to \infty]{} \infty$ or c

Then $\lim \frac{f}{f*g} = \lim \frac{1}{g} \longrightarrow 0$ or $\frac{1}{c}$

$\therefore f = O(f*g)$.

9. Evaluate in an *efficient way* $\sum_{i=4}^{100}(3i^2+5i+4)$

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$$= 3\sum_{i=4}^{100} i^2 + 5\sum_{i=4}^{100} j + \sum_{i=4}^{100} 4$$

$$= 3\left[\sum_{i=1}^{100} i^2 - \sum_{i=1}^{3} i^2\right] + 5\left[\sum_{j=1}^{100} j - \sum_{j=1}^{3} j\right] + 4 * 97.$$

$$= 3\left[\frac{(100)(100+1)(201)}{6} - \frac{3(4)(7)}{6}\right] + 5\left[\frac{100(101)}{2} - \frac{3(4)}{2}\right] + 4*97$$

10. Show that if $x \in \mathbb{R}$, then $x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$

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Let $x = n + d \qquad 0 \leq d < 1$

$$\Rightarrow x - 1 = (n-1) + d$$

$$\lfloor x \rfloor = n$$

$$\lceil x \rceil = n + 1$$

$$x + 1 = (n+1) + d$$

$\Rightarrow$ 1) $x - 1 < \lfloor x \rfloor$

$n - 1 + d = n - (\underbrace{1-d}_{>0}) < n$

2) $\lfloor x \rfloor \leq x$ since $n \leq n + d$

3) $x \leq \lceil x \rceil$ since $n + d \leq n + 1$ because $d<1$

4) $\lceil x \rceil < x + 1$ since $\lceil x \rceil = n + 1 \leq n + d + 1$ ✓