LEBANESE AMERICAN UNIVERSITY
Department of Computer Science and Mathematics
Calculus IV
Exam I Spring 2014 (April 1, 2014)

Name: Mostafa Baghdadi **ID:** 201300681

Question Number	**Grade**
1. 10%	10
2. 8%	8
3. 14%	14
4. 8%	8
5. 15%	15
6. 9%	9
7. 12%	12
8. 12%	12
9.12%	9
Total	97

1. (10%) Show *using the shortest method* that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$.

f

g

the two surface ~~are~~: at each surface there is a tangent plane which have a normal if these two normals are opposite and in the same line so they are tangent

so $\vec{\nabla f} = f_x\vec{i} + f_y\vec{j} + f_z\vec{k} = 6x\vec{i} + 4y\vec{j} + 2z\vec{k}\Big|_{(1,1,2)} = 6\vec{i} + 4\vec{j} + 4\vec{k}$

$\vec{\nabla g} = (2x-8)\vec{i} + (2y-6)\vec{j} + (2z-8)\vec{k}\Big|_{(1,1,2)} =$

$= -6\vec{i} - 4\vec{j} - 4\vec{k}$

they are opposite $\vec{\nabla f} = -\vec{\nabla g}$ So verified

2. (8%) If you are standing on the surface $z = x^2 + \cos(x^2 y)$ at the point $(1, 0, 2)$ and you want your rate of change to be 4, which direction should you move in?

the rate change should be 4.

So that the Directional Derivative $(D_u f)_{P_0} = 4$

So $D_{\vec{u}} f = \vec{\nabla f} \cdot \vec{u} = f_x u_x + f_y u_y = 4$ $\quad u(u_x, u_y)$

$f_x = 2x - 2yx\sin(x^2y) = 2 - 2\sin(0) = 2$ at $(1,0,2)$ $\quad f_x = 2$, $f_y = 0$

$f_y = -x^2\sin(x^2y) = 0$

$f_y u_y = 4$

$u_y = \frac{4}{2} = 2$

So it should move in the direction $\vec{u} = 2\vec{j}$

3. (14%) Let $w(x, y) = 3x^2 - xy$ and let $P(1, 2)$.

(a) Approximate the value of w at $Q(1.01, 1.98)$

Approximate

$w(x,y) = w(x_0,y_0) + w_x(x-x_0) + w_y(y-y_0) = 1 + 4(0.01) - 2(-0.02)$
$= 1.08$

let $x_0 = 1$, $y_0 = 2$ $\quad w(1,2) = 3 - 2 = 1$

* $w_x = 6x - y\Big|_{x_0,y_0} = 6 - 2 = 4$ $\quad$ * $x - x_0 = 1.01 - 1 = 0.01$

* $w_y = -x = -2$ $\quad$ * $y - y_0 = 1.98 - 2 = -0.02$

(b) Find dw and use it to approximate the change in moving from $P(1,2)$ to $Q(1.01, 1.98)$.

~~$dw = w_x \cdot dx$~~ $dw = w_x dx + w_y dy$ $\quad$ $dx = 0.01$ (change in x)

$dy = -0.02$ (change in y)

$= 4(0.01) + (-2)(-0.02)$

$= 0.04 + 0.04 = 0.08$

✓

4. (8%) Find the points on the surface $x^2 - 2y^2 - 4z^2 = 1$ at which the tangent plane is parallel to the plane $4x - 2y + 4z = 5$ (S)

the tagent plane is parallele to the plane $4x-2y+4z=5$ so the normal to the tagent is the same normal to the last plane.

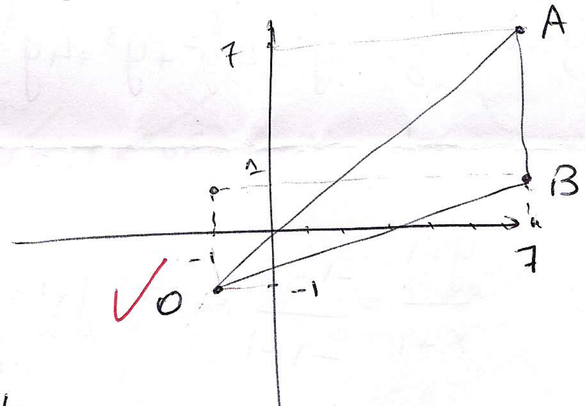
So $\vec{\nabla}_p = 2x\vec{i} - 4y\vec{j} - 8z\vec{k}$ $\quad$ the nomal to the plane (S) is $\vec{N} = (4; -2; 4)$ → next page

$\boxed{\vec{\nabla}_p = \vec{N} \cdot K}$ so $2x = 4K$ $\quad$ $-4y = -2$ $\quad$ $-8z = 4$ $\quad$ the points are $(2K, -1/2K, -1/2K)$

$x = 2K$ $\quad$ $y = -1/2K$ $\quad$ $z = -1/2K$

5. (15%) Let $f(x, y) = x^2 - 4xy + y^3 + 4y$.Find the extreme points of f on the triangular region with vertices $(-1,-1), (7,1)$ and $(7,7)$.

O $\quad$ B $\quad$ A



first.

~~we find~~ we solve $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

$f_x = 2x - 4y = 0 \rightarrow \boxed{x = 2y}$

$f_y = -4x + 3y^2 + 4 = 0$ ✓

$-8y + 3y^2 + 4 = 0 \quad \begin{cases} y = 2 \;;\; x = 4 \\ y = \frac{2}{3} \;;\; x = 4/3 \end{cases}$

for $(4, 2)$

$f_{xx} = 2$ $\quad$ $f_{xy} = -4$ $\quad$ $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ -4 & 12 \end{vmatrix} = 24 - 16 = 8 > 0$

$f_{yy} = 6y = 12$ $\quad$ $f_{xx} > 0$

So $(4,2)$ is minimum

for $(4/3 ; 2/3)$

$f_{xx} = 2$

$f_{yy} = 6y = 6 \cdot \frac{2}{3} = 4$ $\quad$ $\begin{vmatrix} 2 & -4 \\ -4 & 4 \end{vmatrix} = 8 - 16 = -8 < 0$

Saddle point.

the $\vec{\nabla f}$ is $\perp$ to the plane

so $\vec{\nabla f}$ is parallel to the normal of that plane

$\vec{\nabla f} = K . \vec{N}$

$2x\vec{i} - 4y\vec{j} - 8z\vec{k} = K(4\vec{i} - 2\vec{j} + 4\vec{k})$

$2x = 4K \qquad -4y = -2K \qquad -8z = 4K$

$x = 2K \qquad y = \frac{1}{2}K \qquad z = -\frac{1}{2}K$

Points:

$(2K, \frac{1}{2}K, -\frac{1}{2}K)$ K any real non zero number $K \in R^*$

OA: $\frac{y-y_0}{x-x_0} = \frac{y_0 - y_A}{x_0 - x_A}$ $\frac{y+1}{x+1} = \frac{-1-7}{-1-7} \Rightarrow y = x$

on $y = x$ $z = y^2 - 4y^2 + y^3 + 4y = -3y^2 + y^3 + 4y$

$z' = -6y + 3y^2 + 4 = 0 \rightarrow$ never zero. no (maximum)

OB: $\frac{y+1}{x+1} = \frac{-1-7}{-1-1} \Rightarrow y+1 = 4(x+1)$

$\boxed{y = 4x + 5} \Rightarrow x = \frac{y-5}{4}$

on $y = 4x+5$

$\cancel{z = x^2 - 4x(4x+5)}$

$z = (\frac{y-5}{4})^2 - 4(\frac{y-5}{4})y + y^3 + 4y = \frac{y^2}{4} - \frac{10y}{4} + \frac{25}{4} - y^2 + 5y + y^3 + 4y$

$z' = \frac{1}{2}y - \frac{10}{4}y - 2y + 5 + 2y^2 + 4 = 0$

$= -2y - 2y + 9 + 2y^2 = 0$

$-4y + 2y^2 + 9 = 0 \leftrightarrow$ never zero (not maximum)

on AB $\frac{y-7}{x-7} = \frac{7-1}{7-7} \Rightarrow$ $x = 7$

$z = 49 - 28y + y^3 + 4y \rightarrow z' = 3y^2 + 4y - 28 = 0$ $\rightarrow y = 2.4 \quad x = 7$; $\rightarrow y = -3.7 \quad x = 7$

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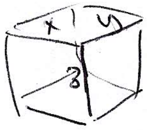
6. (9%) Prove that a closed rectangular box of dimensions x, y, and z of fixed volume V and a minimal surface area is a cube.



surface of a rectangle: $S = 2xy + 2zx + 2yz$ minimal

$V = xyz = c$ (fixed) (constraint)

So using lagrange $\vec{\nabla}_s = S_x\vec{i} + S_y\vec{j} + S_z\vec{k}$

$= (2y+2z)\vec{i} + (2x+2z)\vec{j} + (2x+2y)\vec{k}$ ✓

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$\vec{\nabla}_v = yz\vec{i} + xz\vec{j} + xy\vec{k}$

$\vec{\nabla}_s = \lambda\vec{\nabla}_v$
$$\begin{cases} 2y+2z = \lambda yz \\ 2x+2z = \lambda xz \\ 2x+2y = \lambda xy \\ xyz = c \end{cases}$$

$\dfrac{2y+2z}{y} = \dfrac{2x+2z}{x} \Rightarrow \cancel{2yx} + 2zx = \cancel{2yx} + 2zy$, $x = y$

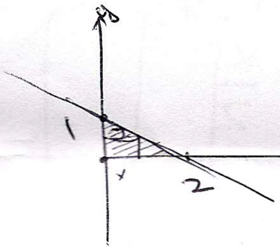
$\dfrac{2x+2z}{z} = \dfrac{2x+2y}{y} \longrightarrow \cancel{2yx} + 2zy = 2xz + \cancel{2yz}$, $x = y$

$\dfrac{2y+2z}{z} = \dfrac{2x+2y}{x}$ $2yx + 2xz = 2xz + 2yz$, $z = x$

So $x = y = x$

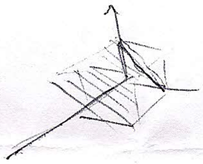
Cube

7. (12%) A rectangle with sides parallel to the axes is inscribed in the region bounded by the axes and the line $x + 2y = 2$. Find the maximum and minimum area of such a rectangle.



$x + 2y = 2$ constant

$A = xy$ function ✓



$A = x.y$

$x + 2y = 2$

$\vec{\nabla}_f = A_x\vec{i} + A_y\vec{j} = y.\vec{i} + x\vec{j}$

Next page →

$\vec{\nabla}_f = \lambda\vec{\nabla}_g$

$y = \lambda x$

$x \cancel{2y} = \lambda g(2)$

$\dfrac{y}{x} = \dfrac{x}{2y}$

$2y^2 = x^2$

$x = \pm\sqrt{y}$

$x + 2y = 2$

$x + 2y = 2$

$\sqrt{y} + 2y = 2$

$\dfrac{y}{x} = \dfrac{x}{2y}$

$4y^2 = x^2$

queshe 5

the basket is

x, y	z	
$(4, 2)$	56	
$(4/3, 2/3)$	-0.1	
$(7, 2.4)$	5.224	
$(7, -3.7)$	87	
$(-1, -1)$	-8	absolute min
$(7, 1)$	320	absolute max
$(7, 7)$		

natural realex

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7 $x + 2y = 2$

$\nabla g = \vec{i} + 2\vec{j}$

$y = \lambda$

$x = 2\lambda$

$x + 2y = 2$

$y = \frac{x}{2}$

minimum area is 0

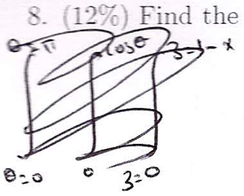
maximum area is 1/2

$x + x = 2$

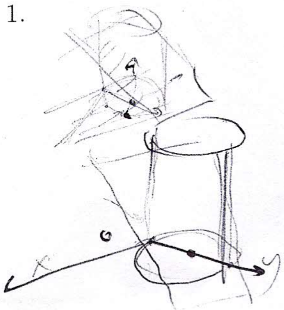
$2x = 2$

$$\boxed{\begin{array}{l} x = 1 \\ y = 1/2 \end{array}}$$

8. (12%) Find the volume of the solid bounded by the cylinder $x = y^2$, $z = 0$, and $x + z = 1$.



$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} dz\, dy\, dx$$

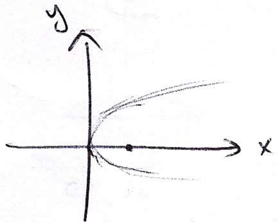


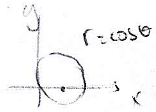
$$\int_{y=0}^{1} \int_{y=\cancel{\sqrt{x}}}^{y=\sqrt{x}} \int_0^{1-x} dz\, dy\, \cancel{dy}$$

OK

$$= \int_0^1 \int_{\sqrt{x}}^{\sqrt{1+x}} 1 - x \; dy\, dx$$

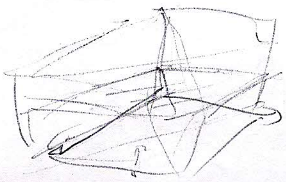
$$= \int_0^1 (1-\sqrt{x})-(1+\sqrt{x}) = \int_0^1 2\sqrt{x} = \frac{1}{3} 2\, x^{1/3} \Big|_0^1 = \frac{2}{3}.$$





9. (12%) Find the volume of the tetrahedron bounded by the four planes, $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

$$\int_0^1 \int_0^{\cancel{1}-x} \int_0^{1-(x+y)} dz\, dx\, dy$$



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