Lebanese American University Spring 2012

Byblos

Calculus IV Date: Apr/03/2012

Test #1 Duration: 2h

Name: N. El Khatib ID:

I. **(30 Points)** Let f be a function defined over $]-\pi\,,\,\pi[$ defined as

$$f(x)=\begin{cases} -\pi-x & \text{if } -\pi<x\le-\frac{\pi}{2}, \\ x & \text{if } -\frac{\pi}{2}\le x<\frac{\pi}{2}, \\ \pi-x & \text{if } \frac{\pi}{2}\le x<\pi. \end{cases}$$

a. Sketch the graphic representation of f and show if f is even or odd.

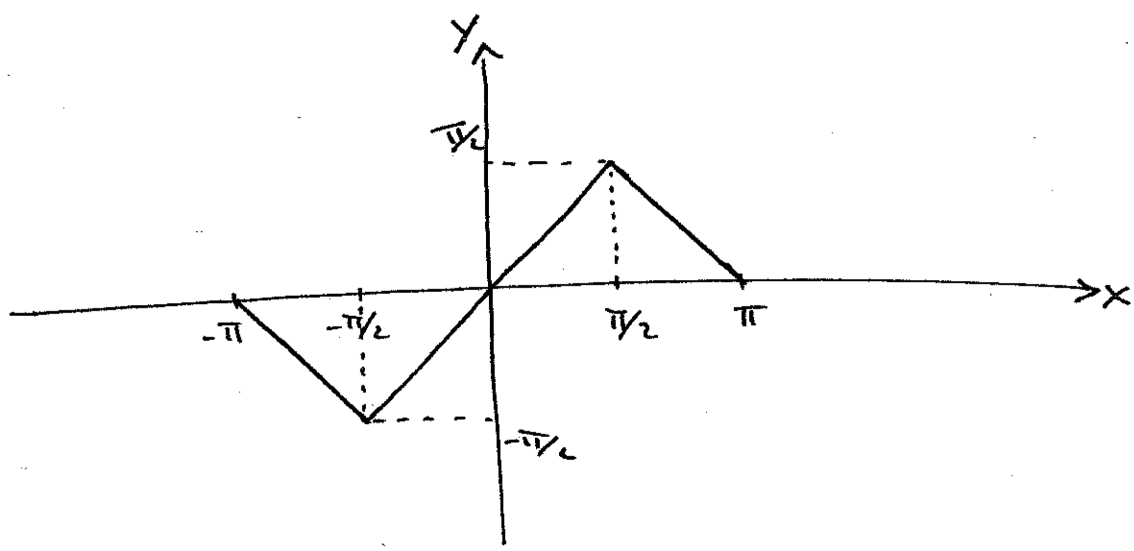
b. Prove that the Fourier series of f is

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)\sin(nx)}{n^2}$$

c. Deduce the values of the sums:

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2},\quad \sum_{n=1}^{\infty}\frac{1}{n^2},\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

a)



Clearly, the function $f(x)$ is odd.

b) $f(x)$ is odd then $a_0=0$ & $a_n=0$.

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx=\frac{2}{\pi}\int_0^{\pi/2}x\sin(nx)\,dx+\frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi-x)\sin(nx)\,dx$$

$$=\frac{2}{\pi}\left[\frac{-x}{n}\cos(nx)+\frac{1}{n^2}\sin(nx)\right]_0^{\frac{\pi}{2}}+\frac{2}{\pi}\left[\frac{-\pi}{n}\cos(nx)+\frac{x}{n}\cos(nx)-\frac{1}{n^2}\sin(nx)\right]_{\pi/2}^{\pi}$$

x	$\sin(nx)$
1	$-\frac{1}{n}\cos(nx)$
0	$-\frac{1}{n^2}\sin(nx)$

$$\frac{2}{\pi}\left(-\cancel{\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)}+\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)-\cancel{\frac{\pi}{n}(-1)^n}+\cancel{\frac{\pi}{n}(-1)^n}+0+\right.$$
$$\left.\cancel{\frac{\pi}{n}\cos\left(\frac{n\pi}{2}\right)}-\cancel{\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)}+\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$=\frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right).$$

Then the Fourie series of f is:

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)\sin(nx)}{n^2}$$

c) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\,?$

at $x=\frac{\pi}{2}$, we have a continuity point then

$$f\left(\frac{\pi}{2}\right)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)}{n^2}=\frac{4}{\pi}\sum\frac{\sin^2\left(\frac{n\pi}{2}\right)}{n^2}$$

but $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$, then

$$\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{n\pi}{2}\right)}{n^2}=\frac{\pi^2}{8}\quad(1)$$

$$\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{n\pi}{2}\right)}{n^2}=\sum_{K=0}^{\infty}\frac{\sin^2\left(\frac{(2K+1)\pi}{2}\right)}{(2K+1)^2}+\underbrace{\cancel{\sum_{K=1}^{\infty}\frac{\sin^2(K\pi)}{(2K)^2}}}_{=0}$$

but $\sin\left(\frac{(2K+1)\pi}{2}\right)=(-1)^K \Rightarrow \sin^2\left(\frac{(2K+1)\pi}{2}\right)=1$

then (1) implies that

$$\boxed{\sum_{K=0}^{\infty}\frac{1}{(2K+1)^2}=\frac{\pi^2}{8}}$$

$$\sum_{n=1} \frac{1}{n^2} = \sum_{n=0} \frac{1}{(2n+1)^2} + \sum_{n=1} \frac{1}{(2n)^2} \iff$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} \iff$$

$$\frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} \iff \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{K=0}^{\infty} \frac{(-1)^{2K+1}}{(2K+1)^2} + \sum_{K=1}^{\infty} \frac{(-1)^{2K}}{(2K)^2}$$

$$= \sum_{K=0}^{\infty} \frac{-1}{(2K+1)^2} + \sum_{K=1}^{\infty} \frac{1}{4K^2}$$

$$= -\sum_{K=0}^{\infty} \frac{1}{(2K+1)^2} + \frac{1}{4}\sum_{K=0}^{\infty} \frac{1}{K^2}$$

$$= -\frac{\pi^2}{8} + \frac{\pi^2}{24}$$

then $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

II. **(30 Points)** For $y \in \mathbb{R}$, consider the function $x = e^{-y^2}$.

a. Find a parametrization for f of the form $\mathbf{r}(t) = f(t)\mathbf{i} + t\mathbf{j}$ with $t \in \mathbb{R}$ and f being a function to be determined. (This parametrization is considered in the questions that follow).

b. Verify that $\mathbf{r}$ is smooth and sketch its graphic representation.

c. Find the unit tangent vector $\mathbf{T}$ at any point of $\mathbf{r}(t)$.

d. Prove that $\forall t \neq 0$ the principal unit normal $\mathbf{N}$ is never parallel to the x-axis.

e. Find the curvature at any point of $\mathbf{r}(t)$.

f. Prove that the curvature is maximum at the point $P(1,0)$ and find the osculating circle at this point.

a) $r(t) = f(t)i + tj$

let $y = t$ $(t \in \mathbb{R})$ then $x = e^{-t^2}$

hence $r(t) = e^{-t^2} i + t j$ ~~with~~ $f(t) = e^{-t^2}$

b) e^{-t^2} is differentiable

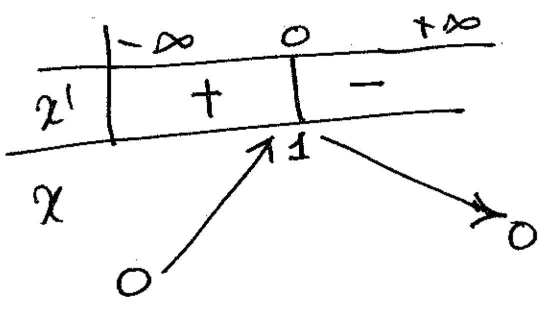
t is differentiable

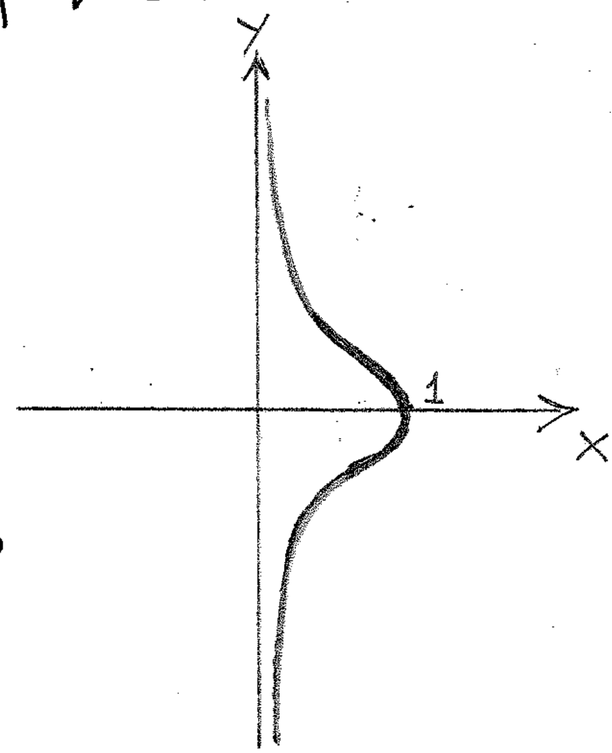
$r'(t) = -2t e^{-t^2} i + j \neq (0,0)$ $\forall t \in \mathbb{R}$

then $r(t)$ is smooth.

$x = e^{-y^2}$ $\qquad x' = -2y e^{-y^2}$

$\lim\limits_{y \to \pm\infty} x = 0$





c) $v(t) = r'(t) = (-2t e^{-t^2}) i + j$

$|v(t)| = \sqrt{4t^2 e^{-2t^2} + 1}$

then $T = \dfrac{v}{|v|} = \dfrac{-2t e^{-t^2}}{\sqrt{1 + 4t^2 e^{-2t^2}}} i + \dfrac{1}{\sqrt{1 + 4t^2 e^{-t^2}}} j$

) The vecta $T = \dfrac{-2te^{-t^2}}{\sqrt{1+4t^2e^{-2t^2}}}\,i + \dfrac{1}{\sqrt{1+4t^2e^{-2t^2}}}\,j$

for $\forall t \neq 0$ the x-component of $T \neq 0$

for $\forall t \neq 0$: y-component of $T \neq 0$

then $\underline{\forall t \neq 0}$ T cannot be parallel to the y-axis

but $N \perp T$ then $\underline{\forall t \neq 0}$ N cannot be parallel to the x-axis.

) $\chi(t) = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$ $\qquad \dot{x} = -2te^{-t^2}, \ \dot{y} = 1$

$\qquad\qquad\qquad\qquad\qquad \ddot{x} = (-2+4t^2)e^{-t^2}, \ \ddot{y} = 0$

then $\chi(t) = \dfrac{|(-2+4t^2)e^{-t^2}|}{(4t^2e^{-2t^2}+1)^{\frac{3}{2}}} = \dfrac{2\,|2t^2-1|\,e^{-t^2}}{(4t^2e^{-2t^2}+1)^{\frac{3}{2}}}$

take $t^2 < \frac{1}{2}$

$$\chi'(t) = \frac{-4t\,e^{-t^2}\left(e^{-2t^2}(-12t^2+6+16t^4)+3-2t^2\right)}{(4t^2e^{-2t^2}+1)^3}$$

$16t^4 - 12t^2 + 6 = (4t^2 - \frac{3}{2})^2 + \frac{15}{4} > 0 \quad \forall t$

&

$3 - 2t^2 \geq 0$ (because we have $t^2 < \frac{1}{2}$)

then $e^{-2t^2}(-12t^2+6+16t^4) + 3 - 2t^2 \geq 0$.

hence the sign of $\chi'(t)$ depends on the sign of "t"

$\chi'(t) = 0$ when $t = 0$.
i.e. at the point $(1,0)$

then χ is max at $(1,0)$

t		t	
$\chi'(t)$	$+$	ϕ	$-$
	$\nearrow$		$\searrow$

r the point $P(1,0)$ $t=0$.

$\chi|_{t=0} = 2$ then the radius of the osculating circle

$= R = \frac{1}{2}$.

$N = -i$ then the circle has the equation:

$$\left.\begin{aligned}&(x-a)^2 + y^2 = \frac{1}{4}\\ &a = 1 - R = \frac{1}{2}\end{aligned}\right\} \Leftrightarrow \boxed{\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}}.$$

III. **(40 Points)** The hyperbola of equation $\frac{x^2}{4} - y^2 = 1$ has two branches. The first branch lies in the side $x \geq 0$ and the second in the side $x \leq 0$. We denote by $\mathcal{H}$ the part that lies in the side $x \geq 0$.

a. Verify that a parametrization for $\mathcal{H}$ can be

$$\mathbf{r}(t) = 2\cosh(t)\mathbf{i} + \sinh(t)\mathbf{j}, \quad t \in \mathbb{R}.$$

Show that $\mathcal{H}$ is smooth and sketch its graphic representation. (The parametrization mentioned above is considered in the questions that follow).

b. Find the velocity vector $\mathbf{v}(t)$ and the unit tangent vector $\mathbf{T}$.

c. Find the curvature κ as a function of t.

d. Prove that κ is maximum at the points $(2, 0)$.

e. Prove that κ does not have a minimum. Give a geometric interpretation of this result.

f. Find the osculating circle at the point $Q(2, 0)$.

g. Generalize the result obtained in d. in order to find at which point of the **whole** hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the curvature κ is maximum.

a) $r(t) = 2\cosh(t) i + \sinh(t) j \quad t \in \mathbb{R}$.

$x(t) = 2\cosh(t) \quad \underline{x > 0}$

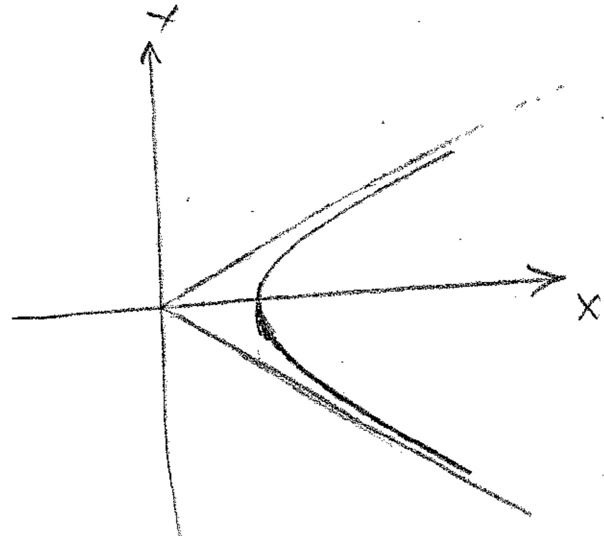
$y(t) = \sinh(t)$

$\frac{x^2}{4} - y^2 = \cosh^2(t) - \sinh^2(t) = 1$. ok✓.

$\cosh(t)$ & $\sinh(t)$ are differentiable.

$r'(t) = 2\sinh(t) i + \cosh(t) j \neq (0,0) \ \forall t$.

then $r(t)$ is smooth.



b) $v(t) = r'(t) = 2\sinh(t) i + \cosh(t) j$

$|v(t)| = \sqrt{4\sinh^2(t) + \cosh^2(t)}$

$= \sqrt{1 + 5\sinh^2(t)}$

$T = \frac{2\sinh(t)}{\sqrt{1+5\sinh^2(t)}} i + \frac{\cosh(t)}{\sqrt{1+5\sinh^2(t)}} j$

c) $\chi = \dfrac{|\ddot{x}\dot{y} - \ddot{y}\dot{x}|}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$

$\dot{x} = 2\sinh(t) \qquad \ddot{x} = 2\cosh(t)$

$\dot{y} = \cosh(t) \qquad \ddot{y} = \sinh(t)$

$$\chi = \frac{|2\cosh^2(t) - 2\sinh^2(t)|}{(1+5\sinh^2(t))^{\frac{3}{2}}} = \frac{2}{(1+5\sinh^2(t))^{\frac{3}{2}}}$$

$\boxed{\max = 2}$

d) χ is max when $1+5\sinh^2(t)$ is min

$\Leftrightarrow \sinh^2(t) = 0 \Leftrightarrow t=0$.

∘ $t=0$ corresponds the point $P(2,0)$.

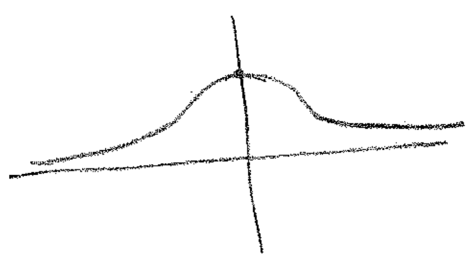
then χ is max on $P(2,0)$.

e) $\chi' = \dfrac{-30\sinh(t)\cosh(t)}{(1+5\sinh^2(t))^{5/2}}$

	$-\infty$		0		$+\infty$
χ'		$+$	0	$-$	
χ	0	$\nearrow$	2	$\searrow$	0

$\lim_{t \to \pm\infty} \chi(t) = 0$

then χ will behave like



hence χ does not have a min.

f) $R = \dfrac{1}{\chi|_{t=0}} = \dfrac{1}{2}$.

$N = +i$

$(x-a)^2 + y^2 = \frac{1}{4} \qquad a = 2 + \frac{1}{2}$.

$(x - \frac{5}{2})^2 + y^2 = \frac{1}{4}$

g) χ is max at the points $(\pm a, 0)$.

IV. **(10 Points)** Consider the planar curve given in polar coordinates by $\rho = \rho(\theta)$, $a < \theta < b$. Show that the curvature is given by the formula

$$\kappa(\theta) = \frac{|2(\rho')^2 - \rho\rho'' + \rho^2|}{[\rho^2 + (\rho')^2]^{\frac{3}{2}}}, \quad \text{where } \rho' = \frac{d\rho}{d\theta}, \text{ and } \rho'' = \frac{d^2\rho}{d\theta^2}.$$

To pass from the polar coordinats to the cartesian coordinate

$$X(\theta) = \rho(\theta)\cos(\theta)$$
$$Y(\theta) = \rho(\theta)\sin(\theta)$$

so we have a parametrization with θ as a parameter.

$$\dot{x} = \rho'\cos(\theta) - \rho\sin(\theta)$$
$$\dot{y} = \rho'\sin\theta + \rho\cos(\theta)$$

$$\ddot{x} = \rho''\cos(\theta) - \rho'\sin\theta - \rho'\sin\theta - \rho\cos(\theta)$$
$$= \rho''\cos(\theta) - 2\rho'\sin(\theta) - \rho\cos(\theta)$$
$$\ddot{y} = \rho''\sin(\theta) + \rho'\cos(\theta) + \rho'\cos(\theta) - \rho\sin\theta$$
$$= \rho''\sin(\theta) + 2\rho'\cos(\theta) - \rho\sin(\theta)$$

we use $\quad \kappa = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$

$$(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}} = \left((\rho'\cos(\theta) - \rho\sin(\theta))^2 + (\rho'\sin(\theta) + \rho\cos\theta)^2\right)^{\frac{3}{2}}$$
$$= \Big(\underline{\rho'^2\cos^2(\theta)} + \underline{\rho^2\sin^2(\theta)} - 2\cancel{\rho'\rho\cos(\theta)\sin(\theta)} +$$
$$\underline{\rho'^2\sin^2(\theta)} + \underline{\rho^2\cos^2(\theta)} + 2\cancel{\rho\rho'\sin(\theta)\cos(\theta)}\Big)^{\frac{3}{2}}$$
$$= (\rho'^2 + \rho^2)^{\frac{3}{2}}.$$

$$|\dot{x}\ddot{y} - \dot{y}\ddot{x}| = \Big|(\rho'\cos(\theta) - \rho\sin(\theta))(\rho''\sin(\theta) + 2\rho'\cos(\theta) - \rho\sin(\theta))$$
$$- (\rho'\sin(\theta) + \rho\cos(\theta))(\rho''\cos(\theta) - 2\rho'\sin(\theta) - \rho\cos(\theta))\Big| =$$

$$\cancel{\rho'\rho''\cos(\theta)}\sin(\theta) + 2\rho'^2\cos^2(\theta) - \cancel{\rho'\rho\cos(\theta)\sin(\theta)} - \rho\rho''\sin^2(\theta)$$

$$-2\rho'\cancel{\rho\sin(\theta)}\cos(\theta) + \rho^2\sin^2(\theta) - \rho'\cancel{\rho''\sin(\theta)}\cos(\theta) + 2\rho'^2\sin^2(\theta)$$

$$+\rho\cancel{\rho'\sin(\theta)}\cos(\theta) - \rho\rho''\cos^2(\theta) + 2\rho\cancel{\rho'\cos(\theta)}\sin(\theta) + \rho^2\cos^2(\theta)\Big|$$

$$= \left|\rho^2 + 2\rho'^2 - \rho\rho''\right|$$

then $$\chi(\theta) = \frac{\left|2\rho'^2 - \rho\rho'' + \rho^2\right|}{\left(\rho'^2 + \rho^2\right)^{\frac{3}{2}}}$$